Q -UNIVERSAL QUASIVARIETIES OF ALGEBRAS

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(Communicated by Louis J. Ratliff, Jr.)

ABSTRACT. A quasivariety of algebras of finite type is Q -universal if its lattice of subquasivarieties has, as a homomorphic image of a sublattice, the lattice of subquasivarieties of any quasivariety of algebras of finite type. A sufficient condition for a quasivariety to be Q -universal is given, thereby adding, amongst others, the quasivarieties of de Morgan algebras, Kleene algebras, distributive p -algebras, distributive double p -algebras, Heyting algebras, double Heyting algebras, lattices containing the modular lattice $M_{3,3}$, MV -algebras, and commutative rings with unity to the known Q -universal quasivarieties.

1. INTRODUCTION

A *quasivariety* is any universal Horn class of algebras or relational structures that contains a trivial algebra or a trivial relational structure, respectively, or, equivalently, any class of similar algebras or relational structures that is closed under isomorphic images, subalgebras, direct products (including direct products of empty families), and ultraproducts (the operations of which are denoted I , S , P , and P_U , respectively). The set of all quasivarieties contained in a given quasivariety $\mathbf{K}$ forms, with respect to inclusion, a lattice denoted $L(\mathbf{K})$.

A quasivariety $\mathbf{K}$ of algebras of finite type is *Q -universal* if, for every quasivariety $\mathbf{M}$ of algebras of finite type, $L(\mathbf{M})$ is a homomorphic image of a sublattice of $L(\mathbf{K})$. It follows immediately that, for any Q -universal quasivariety $\mathbf{K}$, $L(\mathbf{K})$ has a sublattice freely generated by ω elements (hence, satisfies no nontrivial lattice identity) and is of cardinality $2^{\aleph_0}$. The notion was introduced by Sapir in [18] where the first examples of Q -universal quasivarieties were given. In particular, he proved that the quasivariety of commutative 3-nilpotent semigroups is Q -universal and, a fortiori, so too are the quasivarieties of commutative semigroups and all semigroups.

In [7] a sufficient condition (see §2) was given for a quasivariety $\mathbf{K}$ of algebras of finite type to ensure that $L(\mathbf{K})$ fails to satisfy any nontrivial lattice identity. This condition is satisfied by the quasivarieties of de Morgan algebras, Kleene

Received by the editors July 2, 1991 and, in revised form, July 24, 1992.

1991 *Mathematics Subject Classification*. Primary 06B99, 08C15; Secondary 03G25, 06B10.

Key words and phrases. Quasivariety, lattice of quasivarieties, Q -universality, ideal lattice, Fraser-Horn property.

algebras (see [1]), distributive p -algebras, distributive double p -algebras, Heyting algebras, double Heyting algebras (see [7]), and any quasivariety of lattices containing the modular lattice $M_{3,3}$ (see [8]). Since this condition is satisfied by the quasivariety of Heyting algebras, it follows that the quasivarieties of interior algebras (see Blok and Dwinger [3]) and Nelson algebras (see Sendlewski [19]) also satisfy it. The principal aim of this paper, Theorem 3.3, is to prove that this condition is also sufficient for a quasivariety to be Q -universal. In particular, it follows that each of the aforementioned quasivarieties is Q -universal.

A readily obtained consequence of Theorem 3.3 (namely, Corollary 3.4) shows that any quasivariety of finite type whose class of finite algebras satisfies the Fraser-Horn property and contains infinitely many algebras, none of which is embeddable into any other one and each of which is hereditarily simple, is also Q -universal. In particular, it follows, for example, that the quasivariety of Chang's MV -algebras (see [5]) and the quasivariety of commutative rings with unity are also Q -universal.

Section 4 provides one more example of a Q -universal quasivariety. This is a quasivariety $\mathbf{G}$ of algebras of type $(1,1,0)$ for which an explicit description of $L(\mathbf{G})$ was given by Gorbunov in [12] (see also Gorbunov and Tumanov [14]).

As stated above, if a quasivariety $\mathbf{K}$ is Q -universal, then $L(\mathbf{K})$ fails to satisfy every nontrivial lattice identity. Whether the inverse implication is valid remains an open question.

2. PRELIMINARIES

For a set K of similar algebras or relational structures, let $Q(K)$ denote the least quasivariety to contain K . If the algebras of K are indexed, say, by a set I or K has precisely one member A , then we shall sometimes write $Q(I)$ or $Q(A)$, respectively, instead of $Q(K)$. Let $P_{\text{fin}}(\omega)$ denote the lattice of all finite subsets of ω and $P(X)$ denote the lattice of all subsets of a set X , where, for both, set-theoretical union and intersection are the lattice join and meet operations, respectively.

The condition introduced in [7] (see §1) asserts, for a quasivariety $\mathbf{K}$ of finite type, the existence of an infinite family $(A_X : X \in P_{\text{fin}}(\omega))$ of finite algebras satisfying the following postulates:

- (P1) $A_\emptyset$ is a trivial algebra;
- (P2) for $X \in P_{\text{fin}}(\omega)$, if $X = Y \cup Z$, then $A_X \in Q(\{A_Y, A_Z\})$;
- (P3) for $X, Y \in P_{\text{fin}}(\omega)$, if $X \neq \emptyset$ and $A_X \in Q(A_Y)$, then $X = Y$; and
- (P4) for $X \in P_{\text{fin}}(\omega)$, if A_X is a subalgebra of $B \times C$ for finite $B, C \in Q(P_{\text{fin}}(\omega))$, then there are $Y, Z \in P_{\text{fin}}(\omega)$ with $A_Y \in Q(B)$, $A_Z \in Q(C)$, and $X = Y \cup Z$.

3. THE PRINCIPAL RESULT

For a lattice L with a least element 0, let $S(L)$ denote the lattice of all join subsemilattices of L that contain 0. Further, let F denote the free lattice with ω free generators and $I(F)$ its ideal lattice.

Lemma 3.1. *If $\mathbf{K}$ is a quasivariety of algebras of finite type that contains an infinite family of finite algebras satisfying (P1)–(P4), then $I(F)$ is isomorphic to a sublattice of $L(\mathbf{K})$.*

Proof. For $a \in F$, Freese and Nation [11] constructed a finite lattice L_a and a lattice homomorphism $f_a: F \rightarrow S(L_a)$ satisfying

$$(\star) \quad \text{for } b \in F, \text{ if } a \not\leq b, \text{ then } f_a(a) \not\leq f_a(b).$$

The property $(\star)$ allows us to establish the existence of an embedding of $I(F)$ into the direct product $\prod(S(L_a): a \in F)$. Indeed, for $a \in F$, define $g_a: I(F) \rightarrow S(L_a)$ by $g_a(A) = \bigvee f_a(A)$ for $A \in I(F)$. Since $S(L_a)$ is finite, g_a is well defined and preserves lattice meets and joins. Let A and B be distinct elements of $I(F)$, and, with no loss in generality, choose $a \in A \setminus B$. For any $b \in B$, since $a \not\leq b$, it follows that $f_a(a) \not\leq f_a(b)$ by $(\star)$. In particular, $f_a(a) \not\leq g_a(B)$ as $S(L_a)$ is finite. Since $f_a(a) \leq g_a(A)$, it follows that $g_a(A) \neq g_a(B)$. Hence the family $(g_a: a \in F)$ of lattice homomorphisms separates the elements of $I(F)$, which guarantees that $I(F)$ can be embedded in $\prod(S(L_a): a \in F)$.

Obviously, L_a treated as a join semilattice can be embedded in the join semilattice reduct of some $P(X_a)$, where X_a is a finite subset of ω , in such a way that the least element of L_a is mapped to $\emptyset$. This gives that $S(L_a)$ can be embedded in $S(P(X_a))$. Consequently, $I(F)$ is isomorphic to a sublattice of $\prod(S(P(X_a)): a \in F)$. Since F is countable, we may assume that $X_a \cap X_b = \emptyset$ for distinct $a, b \in F$. This assumption allows us to state that a map which assigns to $(S_a: a \in F) \in \prod(S(P(X_a)): a \in F)$ the join subsemilattice of $P_{\text{fin}}(\omega)$ generated by $\bigcup(S_a: a \in F)$ is a lattice embedding of $\prod(S(P(X_a)): a \in F)$ into $S(P_{\text{fin}}(\omega))$. In particular, the lattice $I(F)$ is isomorphic to a sublattice of $S(P_{\text{fin}}(\omega))$.

We now define a mapping $h: L(Q(P_{\text{fin}}(\omega))) \rightarrow S(P_{\text{fin}}(\omega))$ by

$$h(\mathbf{M}) = \{X \in P_{\text{fin}}(\omega): A_X \in \mathbf{M}\},$$

where the A_X 's are members of the infinite family of finite algebras that satisfy (P1)–(P4) as given in the hypothesis of the lemma. By (P1) and (P2), h is well defined. To see that h is onto, for $S \in S(P_{\text{fin}}(\omega))$, let $\mathbf{M} = Q(\{A_X: X \in S\}) = \text{ISPP}_U(\{A_X: X \in S\})$. Clearly, $S \subseteq h(\mathbf{M})$. Consider $A_X \in \mathbf{M}$ where $X \in P_{\text{fin}}(\omega)$. Since A_X is finite of finite type, A_X is a subalgebra of $A_{X_1}^{m_1} \times \cdots \times A_{X_n}^{m_n}$ for some distinct $X_1, \dots, X_n \in S$ and positive integers $m_1, \dots, m_n$. By (P4), there exist $Y, Z \in P_{\text{fin}}(\omega)$ such that $A_Y \in Q(A_{X_1}^{m_1} \times \cdots \times A_{X_{n-1}}^{m_{n-1}}) \subseteq Q(\{A_{X_1}, \dots, A_{X_{n-1}}\})$, $A_Z \in Q(A_{X_n}^{m_n}) \subseteq Q(A_{X_n})$, and $X = Y \cup Z$. By (P3), $Z = X_n$. Further, since $A_Y \in Q(\{A_{X_1}, \dots, A_{X_{n-1}}\})$, it now follows that A_Y is a subalgebra of $A_{X_1}^{p_1} \times \cdots \times A_{X_{n-1}}^{p_{n-1}}$ for some integers $p_1, \dots, p_{n-1}$. Proceeding in this manner, an inductive argument reveals that $X = X_1 \cup \cdots \cup X_n$ and, consequently, $X \in S$, as required. Clearly, h preserves arbitrary meets. It is also the case that h preserves arbitrary joins. To see this, it is enough to show that if $X \in h(\bigvee(\mathbf{M}_i: i \in I))$, then $X \in \bigvee(h(\mathbf{M}_i): i \in I)$. If $X \in h(\bigvee(\mathbf{M}_i: i \in I))$, then A_X is a subalgebra of $A_{X_1}^{m_1} \times \cdots \times A_{X_n}^{m_n}$ for some $A_{X_1}, \dots, A_{X_n} \in \bigcup(\mathbf{M}_i: i \in I)$ where, arguing as above, $X = X_1 \cup \cdots \cup X_n$. By (P2), $X \in \bigvee(h(\mathbf{M}_i): i \in I)$. In summary, h is a complete lattice epimorphism. Since h is a complete lattice epimorphism, it is a bounded lattice epimorphism; that is, each congruence class of $\text{Ker}(h)$ has both a least and a greatest element.

Lemma 3 of Davey and Sands [6] states that, for lattices A, B and L , if A is complete and L satisfies Whitman's condition, then, for each bounded

lattice epimorphism $\varphi: A \rightarrow B$ and lattice homomorphism $\psi: L \rightarrow B$, there exists a lattice homomorphism $\delta: L \rightarrow A$ such that $\varphi \circ \delta = \psi$. In the present context, $L(Q(P_{\text{fin}}(\omega)))$ is a complete lattice; $I(F)$ satisfies Whitman's condition as shown by Baker and Hales [2], $h: L(Q(P_{\text{fin}}(\omega))) \rightarrow S(P_{\text{fin}}(\omega))$ is a bounded lattice epimorphism as just shown, and $I(F)$ is isomorphic to a sublattice of $S(P_{\text{fin}}(\omega))$ as shown above. It follows that $I(F)$ is isomorphic to a sublattice of $L(Q(P_{\text{fin}}(\omega)))$, and, consequently, $I(F)$ is isomorphic to a sublattice of $L(\mathbf{K})$, as required. $\square$

Lemma 3.2 (cf. Grätzer [15, p. 80]). *Every algebraic lattice with countably many compact elements is a homomorphic image of $I(F)$.*

Proof. Let L be an algebraic lattice, and let the set C of compact elements of L be countable. The sublattice D of L generated by C is countable, and, therefore, it is a homomorphic image of F , say, by a homomorphism g . Hence, the lattice $I(D)$ of ideals of D is a homomorphic image of $I(F)$ by virtue of the mapping $I \mapsto g(I)$. Consequently, using the fact that L is compactly generated, the lattice $I(C)$ of ideals of C (where C is regarded as a join semilattice with 0) is a homomorphic image of $I(D)$ under the mapping $I \mapsto I \cap C$. Since $I(C) \cong L$, it follows that L is a homomorphic image of $I(F)$. $\square$

Theorem 3.3. *Let $\mathbf{K}$ be a quasivariety of algebras of finite type that contains an infinite family of finite algebras satisfying (P1)–(P4). For every quasivariety $\mathbf{M}$ of algebras of finite type, $L(\mathbf{M})$ and its dual lattice $L^d(\mathbf{M})$ are homomorphic images of a sublattice of $L(\mathbf{K})$. In particular, $\mathbf{K}$ is Q -universal.*

Proof. $L^d(\mathbf{M})$ is an algebraic lattice, and, since the type of $\mathbf{M}$ is finite, it has countably many compact elements. Therefore, by Lemmas 3.1 and 3.2, $L^d(\mathbf{M})$ is a homomorphic image of a sublattice of $L(\mathbf{K})$.

To show that $L(\mathbf{M})$ is a homomorphic image of a sublattice of $L(\mathbf{K})$, for $A \in \mathbf{M}$, let $R(A)$ denote the relational structure obtained from A as follows. $R(A)$ retains the same universe as A , but each operation f of A is replaced by a relation r with arity one more than that of f where, for $a_1, \dots, a_{n+1} \in A$, $r(a_1, \dots, a_{n+1})$ holds if and only if $f(a_1, \dots, a_n) = a_{n+1}$. Let $R(\mathbf{M})$ denote the class of all relational structures that are isomorphic to substructures of the relational structures of the form $R(A)$ for $A \in \mathbf{M}$. Clearly, $R(\mathbf{M})$ is a quasivariety. Since $R(\mathbf{M})$ is locally finite and the type of $R(\mathbf{M})$ is finite, the lattice $L(R(\mathbf{M}))$ of quasivarieties contained in $R(\mathbf{M})$ is algebraic and has only countably many compact elements. Hence, by Lemma 3.2, $L(R(\mathbf{M}))$ is a homomorphic image of $I(F)$ which, by Lemma 3.1, is isomorphic to a sublattice of $L(\mathbf{K})$.

To complete the proof, it remains to show that $L(\mathbf{M})$ is a homomorphic image of $L(R(\mathbf{M}))$. Define $h: L(R(\mathbf{M})) \rightarrow L(\mathbf{M})$ by

$$h(N) = \{A \in \mathbf{M}: R(A) \in N\}.$$

Clearly, h is well defined, onto, and preserves meets. To see that h also preserves joins, let $A \in h(N_0 \vee N_1)$. Thus, $A \in \mathbf{M}$, $R(A) \in N_0 \vee N_1 = \text{ISP}(N_0 \vee N_1)$, and, consequently, $R(A)$ is isomorphic to a subdirect product of some $A_i \in N_i$ where $i < 2$. For $i < 2$, since A_i is a homomorphic image of $R(A)$ and $A_i \in R(\mathbf{M})$, it follows that $A_i = R(B_i)$ for some $B_i \in \mathbf{M}$. In particular, $B_i \in h(N_i)$, and A is a subdirect product of B_0 and B_1 . Hence,

$A \in h(\mathbf{N}_0) \vee h(\mathbf{N}_1)$, and $h(\mathbf{N}_0 \vee \mathbf{N}_1) \leq h(\mathbf{N}_0) \vee h(\mathbf{N}_1)$. Thus, $h(\mathbf{N}_0 \vee \mathbf{N}_1) = h(\mathbf{N}_0) \vee h(\mathbf{N}_1)$, as required. $\square$

In the above proof we referred to the fact that the lattice $L(R(\mathbf{M}))$ is algebraic, which in turn was guaranteed by the assumption that the type of $\mathbf{M}$, and thus the type of $R(\mathbf{M})$, is finite. It turns out, however, that without this assumption the lattice $L(R(\mathbf{M}))$ may not be algebraic. To see this, let $\mathbf{M}$ be the quasivariety consisting of all algebras of the form $(A; f_i)_{i < \omega}$, where each f_i is a unary operation. Let $\mathbf{N}$ denote the class of all 1-element relational structures that belong to $R(\mathbf{M})$, and, for each $X \subseteq \omega$, let B_X denote the relational structure $(\{a\}; r_i)_{i < \omega}$, where a is a fixed element, r_i is a 2-ary relation, and $r_i(a, a)$ holds in B_X if and only if $i \in X$. Notice that $\mathbf{N}$ is a quasivariety and each B_X belongs to $\mathbf{N}$. Now, fix $Y_0 \supset Y_1 \supset Y_2 \supset \dots$ an infinite descending chain of subsets of ω , and let Y denote their intersection. Since B_Y is isomorphic to $\prod(B_{Y_i}; i < \omega)$, $Q(B_Y) \leq \bigvee(Q(B_{Y_i}); i < \omega)$. However, $\bigvee(Q(B_{Y_i}); i < n) = I(\{B_{Y_0}, \dots, B_{Y_{n-1}}, B_\omega\})$ where $n < \omega$, so, $Q(B_Y) \not\leq \bigvee(Q(B_{Y_i}); i < n)$ for all $n < \omega$. Since $Q(B_Y) (= I(\{B_Y, B_\omega\}))$ is an atom in $L(\mathbf{N})$, it follows that $L(\mathbf{N})$ is not algebraic, and since $L(\mathbf{N})$ forms a principal ideal in $L(R(\mathbf{M}))$, neither is the lattice $L(R(\mathbf{M}))$. The above arguments that $L(\mathbf{N})$ is not algebraic are taken from Gorbunov and Tumanov [13].

A class $\mathbf{K}$ of algebras satisfies the *Fraser-Horn property* providing, for all $A_0, A_1 \in \mathbf{K}$, whenever Θ is a congruence relation on $A_0 \times A_1$, there exist congruence relations Θ_0 and Θ_1 on A_0 and A_1 , respectively, such that Θ is of the form $\Theta_0 \times \Theta_1$ (see Fraser and Horn [10]).

Corollary 3.4. *If $\mathbf{K}$ is a quasivariety of algebras of finite type whose class of finite algebras satisfies the Fraser-Horn property and contains infinitely many algebras, none of which is embeddable into any other one and each of which is hereditarily simple, then $\mathbf{K}$ is Q -universal.*

Proof. Let $(A_i; i < \omega)$ be a family of finite algebras in $\mathbf{K}$, no two of which are embeddable into one another and each of which is hereditarily simple. Set $A_X = \prod(A_i; i \in X)$ for each $X \in P_{\text{fin}}(\omega)$. It is not too hard to see that the family $(A_X; X \in P_{\text{fin}}(\omega))$ fulfills (P1)–(P4), and, consequently, $\mathbf{K}$ is Q -universal. $\square$

For one application of the above corollary, consider the class $\mathbf{W}$ of Wajsberg algebras whose termwise definitionally equivalent variant, called MV -algebras, was studied by Chang [5] (see also Komori [16], Mundici [17], and Font, Rodríguez, and Torrens [9]). The term “Wajsberg algebra” is due to Font, Rodríguez, and Torrens [9]. The class of Wajsberg algebras forms a variety, and, thus, it also forms a quasivariety. Moreover, since $\mathbf{W}$ is a congruence distributive variety, it satisfies the Fraser-Horn property. Let $W_n = (\{0, \dots, n-1\}; \rightarrow, \neg, 0)$, where $x \rightarrow y = \max(y-x, 0)$ and $\neg x = n-(x+1)$. It follows that W_n is a Wajsberg algebra that is simple which, for $n-1$ a prime, has no proper subalgebras. Applying Corollary 3.4, yields that $\mathbf{W}$ is Q -universal.

Since the variety $\mathbf{W}$ is termwise definitionally equivalent to the variety of MV -algebras and also to some variety of bounded hoops (see Blok and Pigozzi [4]), it follows that the variety of MV -algebras and the variety of all bounded hoops are also Q -universal.

For another application of Corollary 3.4, consider the variety $\mathbf{R}$ of commutative rings with unity in signature. Clearly $\mathbf{R}$ satisfies the Fraser-Horn property, and the family Z_p , where p is a prime number, of rings of integers modulo p with unity in signature fulfills the assumptions of Corollary 3.4. Thus $\mathbf{R}$ is Q -universal as well.

4. ANOTHER Q -UNIVERSAL QUASIVARIETY

We conclude with one more example of a Q -universal quasivariety.

Let $\mathbf{G}$ denote the quasivariety of all algebras $(A; f, g, e)$ of type $(1, 1, 0)$ which satisfies the following quasi-identities:

$$fg(x) = x, \quad gf(x) = x,$$

and

$$(f(x) = x \ \& \ g(x) = x) \Leftrightarrow x = e.$$

Let N denote the lattice defined on ω whose ordering relation $\leq$ is given by $0 \leq 0$ and, for $m \neq 0$, $n \leq m$ if m divides n (that is to say, the lattice of ideals of the ring of integers). Further, let $S_\wedge(N)$ denote the lattice of all complete meet subsemilattices of N . It was shown by Gorbunov [12] (see also Gorbunov and Tumanov [14]) that $L(\mathbf{G}) \cong S_\wedge(N)$.

Theorem 4.1. $\mathbf{G}$ is a Q -universal quasivariety.

Proof. Let $(X_n: n \geq 1)$ be a fixed partition of the set of prime numbers such that $|X_n| = n$. Define a mapping $f_n: P(X_n) \rightarrow \omega$ by $f_n(\emptyset) = 1$ and $f_n(X) = \prod X$ if $X \neq \emptyset$. Since f_n is one-to-one, $f_n(\emptyset) = 1$, so $f_n(X \cup Y) = f_n(X) \wedge f_n(Y)$ for all $X, Y \in P(X_n)$, the lattice $S(P(X_n))$ is isomorphic to a sublattice of $S_\wedge(N)$. Furthermore, since $(X_n: n \geq 1)$ is a partition, it follows that $\prod(S(P(X_n)): n \geq 1)$ is isomorphic to a sublattice of $S_\wedge(N)$. As argued in the proof of Lemma 3.1, $I(F)$ is isomorphic to a sublattice of $\prod(S(P(X_n)): n \geq 1)$, and, hence, $I(F)$ can be embedded in $S_\wedge(N)$. As argued in the proof of Theorem 3.3, if $\mathbf{M}$ is a quasivariety of algebras of finite type, then $L(\mathbf{M})$ is a homomorphic image of $I(F)$. Since $S_\wedge(N) \cong L(\mathbf{G})$, it follows that $\mathbf{G}$ is Q -universal. $\square$

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