THE EXISTENCE QUESTION IN THE CALCULUS OF VARIATIONS: A DENSITY RESULT

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ABSTRACT. We show the existence of a dense subset $\mathcal{D}$ of $\mathcal{E}(\mathbb{R})$ such that, for g in it, the problem

$$\text{minimum } \int_0^T g(x(t)) dt + \int_0^T h(x'(t)) dt, \quad x(0) = a, \quad x(T) = b$$

admits a solution for every lower semicontinuous h satisfying growth conditions

INTRODUCTION

The first basic aim of the Calculus of Variations is the problem of yielding sufficient conditions for the existence of solutions to the classical minimum problem

$$(P) \quad \text{Minimize } \int_0^T f(t, x(t), x'(t)) dt : x(0) = a, \quad x(T) = b$$

on a suitable space of functions x . The traditional answer to the sufficiency has been to impose that the map $x' \mapsto f(t, x, x')$ be convex (see [Ce]); more recently, in an effort to provide existence criteria other than convexity in x' , some special sufficient condition has been given. More precisely, for the class of maps of the form $f(t, x, x') = g(t, x) + h(t, x')$, existence of solutions has been obtained by requiring that the map $x \mapsto g(t, x)$ be monotonic (for x in $\mathbb{R}$) [M] or, for x in $\mathbb{R}^n$, that the same map be concave [C-C]. Hence, for maps of the form $g(x) + h(x')$ the property of yielding existence of solutions so far seems to belong to a very narrow and special class of functions: those that are either convex in x' or concave or monotonic in x . Existence seems to be a property related to very special geometric behaviour in x or in x' . The purpose of this note is to show that (for x in $\mathbb{R}$) this is not so. We consider the class of functions $g(x) + h(x')$ where $g : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and h is lower semicontinuous; we show that there exists a subset $\mathcal{D}$ of the space of continuous functions, dense for the topology of uniform convergence

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on compacta, such that, for g in $\mathcal{D}$, problem (P) has existence of solutions for every function h satisfying the usual growth conditions.

NOTATION AND PRELIMINARY RESULTS

Let us denote by $\partial f(x)$ the subdifferential of a function f at x . Also, by $\text{extr}(S)$ we mean the set of extreme points of S . The following lemma will be used later.

Lemma 1. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be convex, l.s.c., $f(x') \geq \psi(|x'|) + \gamma$ ($\gamma \in \mathbb{R}$, $\psi \geq 0$ convex, l.s.c. and $\lim_{r \rightarrow \infty} \psi(r)/r = +\infty$). Then, for each measurable $x' : I = [\alpha, \beta] \rightarrow \mathbb{R}$ with the property that $t \mapsto f(x'(t)) \in L^1$, there exists an integrable selection ω of $\Phi(t) : t \mapsto (\partial f)^{-1}(\partial f(x'(t)))$ with values in $\text{extr}(\Phi(t))$ and such that*

$$\int_I \omega(t) dt = \int_I x'(t) dt, \quad \int_I f(\omega(t)) dt = \int_I f(x'(t)) dt$$

and, for each t ,

$$\int_{\alpha}^t \omega(s) ds \leq \int_{\alpha}^t x'(s) ds.$$

Proof. Let us first remark that each inverse image under ∂f of a point $c \in \mathbb{R}$ is closed, convex (f being convex), and bounded, since $\lim_{x' \rightarrow \infty} f(x')/|x'| = +\infty$. It follows that each $(\partial f)^{-1}(c)$ is either empty or a closed interval. Let $(c_i)_{i \in J \subset \mathbb{N}}$ be the values of ∂f whose inverse image under ∂f is a nontrivial interval $[a_i, b_i]$ ($a_i \neq b_i$), and set $K_i = (x')^{-1}([a_i, b_i])$ for i in J , $K_0 = I \setminus \bigcup_i K_i$. Then, by Lemma 3.4 in [A-C], there exists a measurable selection σ_i of the set-valued map $\Phi(t)\chi_{K_i}(t)$ with values in $\text{extr}(\Phi(t)\chi_{K_i}(t)) = \{a_i, b_i\}$ such that

$$(1) \quad \int_I \sigma_i(t)\chi_{K_i}(t) dt = \int_I x'(t)\chi_{K_i}(t) dt$$

and, for each t ,

$$(2) \quad \int_{\alpha}^t \sigma_i(s)\chi_{K_i}(s) ds \leq \int_{\alpha}^t x'(s)\chi_{K_i}(s) ds.$$

For each $i \in J$, we have

$$\int_I f(x'(t))\chi_{K_i}(t) dt = \int_I f(a_i) + c_i(x'(t) - a_i)\chi_{K_i}(t) dt;$$

hence, by (1)

$$(3) \quad \begin{aligned} \int_I f(x'(t))\chi_{K_i}(t) dt &= \int_I f(a_i) + c_i(\sigma_i(t) - a_i)\chi_{K_i}(t) dt \\ &= \int_I f(\sigma_i(t))\chi_{K_i}(t) dt \end{aligned}$$

since $\sigma_i(t) \in \{a_i, b_i\}$. It follows that, for each $n \in \mathbb{N}$,

$$\int_I \psi \left(|x'| \chi_{K_0} + \sum_{i \leq n} |\sigma_i| \chi_{K_i} \right) dt \leq \int_I f(x'(t)) dt - \int_I \gamma dt;$$

i.e., the functions $\omega_n = x' \chi_{K_0} + \sum_{i \leq n} \sigma_i \chi_{K_i}$, $n \in \mathbb{N}$, are equi-integrable. Set $\omega(t) = x' \chi_{K_0}(t) + \sum_i \sigma_i(t) \chi_{K_i}(t) = \lim_n \omega_n(t)$. Vitali's convergence theorem [E-T, VIII, Corollary 1.3] yields that ω is in L^1 . By (1), (2), (3), ω has the required properties. $\square$

Definition. $\mathcal{D}$ is the subset of $\mathcal{E}(\mathbb{R})$ of those piecewise affine functions g such that: (i) any interval in which g is affine with a given nonzero slope is bounded; (ii) adjacent to any such maximal interval, on both the left and the right, is one in which g is constant; and (iii) any bounded set contains only finitely many such intervals.

Proposition 1. $\mathcal{D}$ is dense in $\mathcal{E}(\mathbb{R})$.

MAIN RESULT

Consider the following problem:

Problem P.

$$\text{Minimize } \int_0^T g(x(t)) dt + \int_0^T h(x'(t)) dt$$

on the subset of $W^{1,p}([0, T], \mathbb{R})$ of those functions satisfying the boundary conditions $x(0) = a$, $x(T) = b$.

We shall consider the following

Growth condition G. If $p = 1$, there exist a convex l.s.c. function ψ such that $\lim_{r \rightarrow \infty} \psi(r)/r = +\infty$, $\gamma \in \mathbb{R}$, and $h(x') \geq \psi(|x'|) + \gamma$; if $p > 1$, there exist $\sigma > 0$ (β/σ being strictly less than the best Sobolev constant), and $h(x') \geq \sigma|x'|^p + \gamma$.

We have the following

Theorem 1. Let $p \geq 1$ and $g \in \mathcal{D}$ be such that $g(x) \geq \alpha - \beta|x|^p$ for every x ($\alpha, \beta \in \mathbb{R}$). Let h satisfy the growth condition (G). Then, Problem (P) admits at least a solution.

Proof. Let $\tilde{x}$ be a solution to the relaxed problem associated to (P), and set

$$\Delta_1 = \min\{\tilde{x}(t) : t \in [0, T]\}, \quad \Delta_2 = \max\{\tilde{x}(t) : t \in [0, T]\}.$$

Let d_1 be the greater discontinuity point of g' less than or equal to Δ_1 , $d_2 < \dots < d_{n-1}$ be those inside $]-\Delta_1, \Delta_2[$, d_n be the next after d_{n-1} , and set

$$\epsilon = \frac{1}{2} \min\{|d_{i+1} - d_i| : i = 1, \dots, n-1\}.$$

Claim (a). We claim that $[0, T]$ can be partitioned in a countable union of disjoint intervals I_j ($j \in \mathbb{N}$) such that g is monotonic on $\tilde{x}(I_j)$.

Proof of the claim. Consider the three sets A , V , B defined by

$$A = \bigcup_{i=1}^{n-1}]d_i + \epsilon, d_{i+1} - \epsilon[, \quad V = \{d_i - \epsilon, d_i + \epsilon : i = 1, \dots, n\}$$

$$B = \bigcup_{i=1}^n]d_i - 2\epsilon, d_i + 2\epsilon[.$$

By the continuity of $\tilde{x}$, the inverse image of A under $\tilde{x}$ is a countable union of disjoint relatively open subintervals (σ_i, τ_i) of $[0, T]$. The image of each subinterval is contained in one of the open intervals $]d_i + \epsilon, d_{i+1} - \epsilon[$ on which

g is affine. By the continuity of $\tilde{x}$, there exists δ such that $|t - s| < \delta \Rightarrow |\tilde{x}(t) - \tilde{x}(s)| < \epsilon$. Consider those subintervals (σ_i, τ_i) whose diameter $\tau_i - \sigma_i \geq \delta$, say for $i = 1, \dots, m$. These are the first elements of our partition. Again by continuity, for each i , at least one between $\tilde{x}(\sigma_i)$ and $\tilde{x}(\tau_i)$ is in V (actually both, except for the case $\sigma_i = 0$ or $\tau_i = T$). Consider the finite union of closed subintervals of $[0, T]$ that is the complement of the finite union of open subintervals (σ_i, τ_i) , $i = 1, \dots, m$; they are the intervals $[0, \sigma_1], \dots, [\tau_{m-1}, \sigma_m], [\tau_m, T]$. For t in this complement, $\tilde{x}(t)$ is in B . In fact, either t is in $\tilde{x}^{-1}(A) \setminus \bigcup_{i=1}^m (\sigma_i, \tau_i)$ or $\tilde{x}(t)$ is in $\bigcup_i [d_i - \epsilon, d_i + \epsilon]$.

In the first case, from the choice of δ and the remark on the behaviour at the extremes of the intervals, there exists d_i such that $|\tilde{x}(t) - d_i| < 2\epsilon$, i.e., $\tilde{x}(t) \in B$.

The second case holds since each $[d_i - \epsilon, d_i + \epsilon]$ is in B . Since B is open, its counter image $\tilde{x}^{-1}(B)$ is a countable collection of open subintervals. Consider the image of any such subinterval: it is contained in one (and only one) $]d_i - 2\epsilon, d_i + 2\epsilon[$. On one of $]d_i - 2\epsilon, d_i[$ or $]d_i, d_i + 2\epsilon[$ g is constant, on the other affine; g is monotonic on $]d_i - 2\epsilon, d_i + 2\epsilon[$. Intersect each subinterval with the finite collection of intervals $[\tau_i, \sigma_{i+1}] : i = 0, \dots, m$. The union of this countable collection of intervals and of $(\sigma_i, \tau_i) : i = 1, \dots, m$ is the required partition of $[0, T]$.

Claim (b). By [E-T, Proposition IX.3.1] in the case $p > 1$ and by a slight modification of the same reasoning in the case $p = 1$ under the "superlinearity" growth condition (G), there exist measurable $p_1, p_2, v_1, v_2 : [0, T] \rightarrow \mathbb{R}$ ($p_i \geq 0$, $p_1 + p_2 = 1$) such that

$$\begin{cases} \tilde{x}' = p_1 v_1 + p_2 v_2 & \text{a.e.}, \\ h^{**}(\tilde{x}'(t)) = p_1(t)h(v_1(t)) + p_2(t)h(v_2(t)) & \text{a.e.}; \end{cases}$$

hence, $\text{extr}(\partial h^{**})^{-1}(\partial h^{**}(\tilde{x}'(t))) = \{v_1(t), v_2(t)\}$ a.e. Let $I_j = (\alpha_j, \beta_j)$ be one of the intervals considered in (a). Let us explicitly carry out the construction for the case $g' < 0$ on $]d_{i-1}, d_i[$; the other cases are treated similarly. We claim that there exists a measurable partition E_1^j, E_2^j of $[\alpha_j, \beta_j]$ such that:

- (i) $v_1 \chi_{E_1^j} + v_2 \chi_{E_2^j} \in L^p(\alpha_j, \beta_j)$;
- (ii) $\int_{\alpha_j}^{\beta_j} v_1 \chi_{E_1^j} + v_2 \chi_{E_2^j} dt = \int_{\alpha_j}^{\beta_j} p_1 v_1 + p_2 v_2 dt$;
- (iii) $\int_{\alpha_j}^t v_1 \chi_{E_1^j} + v_2 \chi_{E_2^j} ds \geq \int_{\alpha_j}^t p_1 v_1 + p_2 v_2 ds$ for each $t \in (\alpha_j, \beta_j)$;
- (iv) $\int_{\alpha_j}^{\beta_j} p_1(t)h(v_1(t)) + p_2(t)h(v_2(t)) dt$
 $= \int_{\alpha_j}^{\beta_j} h(v_1(t))\chi_{E_1^j}(t) + h(v_2(t))\chi_{E_2^j}(t) dt$;
- (v) $|\int_{\alpha_j}^t v_1 \chi_{E_1^j} + v_2 \chi_{E_2^j} ds - \int_{\alpha_j}^t p_1 v_1 + p_2 v_2 ds| < \epsilon$ for each $t \in (\alpha_j, \beta_j)$.

Proof of the claim. Set $\psi(x') = |x'|^p$ if $p > 1$. Let us first remark that by La Vallée-Poussin's Theorem, the set

$$\mathcal{H} = \left\{ f \in L^1(\alpha_j, \beta_j) : \int_{\alpha_j}^{\beta_j} \psi(|f|) \leq \int_{\alpha_j}^{\beta_j} h^{**}(\tilde{x}'(t)) dt - \int_{\alpha_j}^{\beta_j} \gamma dt \right\}$$

is equi-integrable. Let $\rho > 0$ be such that $\int_A |f| dt < \epsilon/2$ for each measurable subset A of I_j whose measure is less than ρ and for each $f \in \mathcal{H}$. Let $\alpha_j = \gamma_0 < \gamma_1 < \dots < \gamma_m = \beta_j$ be a subdivision of I_j such that $\max_i |\gamma_{i+1} - \gamma_i| < \rho$. By Lemma 1 and the above remark on the inverse image of the subdifferential of

h^{**} , for each interval $[\gamma_k, \gamma_{k+1}]$, there exists a measurable partition $E_{1,k}, E_{2,k}$ satisfying:

- (i)_k $v_1 \chi_{E_{1,k}} + v_2 \chi_{E_{2,k}} \in L^p(\gamma_k, \gamma_{k+1})$;
- (ii)_k $\int_{\gamma_k}^{\gamma_{k+1}} v_1 \chi_{E_{1,k}} + v_2 \chi_{E_{2,k}} dt = \int_{\gamma_k}^{\gamma_{k+1}} p_1 v_1 + p_2 v_2 dt$;
- (iii)_k $\int_{\gamma_k}^t v_1 \chi_{E_{1,k}} + v_2 \chi_{E_{2,k}} ds \geq \int_{\gamma_k}^t p_1 v_1 + p_2 v_2 ds$ for each $t \in (\gamma_k, \gamma_{k+1})$;
- (iv)_k $\int_{\gamma_k}^{\gamma_{k+1}} \sum_{i=1}^2 p_i(t) h(v_i(t)) dt = \int_{\gamma_k}^{\gamma_{k+1}} \sum_{i=1}^2 h(v_i(t)) \chi_{E_{i,k}}(t) dt$.

Set $E_1^j = \bigcup_{k=0}^{m-1} E_{1,k}$, $E_2^j = \bigcup_{k=0}^{m-1} E_{2,k}$. Then (i), (ii), (iii), (iv) can be trivially deduced from their corresponding (i)_k, (ii)_k, (iii)_k, (iv)_k. In order to prove (v), fix $t \in (\alpha_j, \beta_j)$ and let k be such that $t \in [\gamma_k, \gamma_{k+1}]$. Let us write that

$$(4) \quad \int_{\alpha_j}^t \sum_i (p_i - \chi_{E_i^j}) v_i ds = \int_{\alpha_j}^{\gamma_k} \sum_i (p_i - \chi_{E_i^j}) v_i dt + \int_{\gamma_k}^t \sum_i (p_i - \chi_{E_i^j}) v_i ds.$$

By (ii)_k, the first term of the right-hand side of the above equality is zero. Furthermore, we have

$$\psi(|v_1| \chi_{E_1^j} + |v_2| \chi_{E_2^j}) \leq h(v_1(t)) \chi_{E_1^j}(t) + h(v_2(t)) \chi_{E_2^j}(t) - \gamma$$

so that, by (iv)_k, $v_1 \chi_{E_1^j} + v_2 \chi_{E_2^j} \in \mathcal{H}$. Let us recall that $|t - \gamma_k| < \rho$; hence, by equi-integrability we have

$$\left| \int_{\alpha_j}^t \sum_i (p_i - \chi_{E_i^j}) v_i ds \right| \leq \int_{\gamma_k}^t p_1 |v_1| + p_2 |v_2| ds + \int_{\gamma_k}^t |v_1| \chi_{E_1^j} + |v_2| \chi_{E_2^j} ds \\ < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which proves the claim.

Claim (c). Let us denote by $\bar{x}_j : [\alpha_j, \beta_j] = I_j \rightarrow \mathbb{R}$ the function defined by

$$\bar{x}_j(t) = \tilde{x}(\alpha_j) + \int_{\alpha_j}^t v_1 \chi_{E_1^j} + v_2 \chi_{E_2^j} ds.$$

Then, by (i), $\bar{x}_j' \in L^p(\alpha_j, \beta_j)$, and by (ii) we have $\bar{x}_j(\beta_j) = \tilde{x}(\beta_j)$. Furthermore, by (iv),

$$(5) \quad \int_{\alpha_j}^{\beta_j} h^{**}(\bar{x}_j'(t)) dt = \int_{\alpha_j}^{\beta_j} h(\bar{x}_j'(t)) dt.$$

Since, by definition, $\bar{x}_j(\alpha_j) = \tilde{x}(\alpha_j)$ then, by (iii), $\bar{x}_j(t) \geq \tilde{x}(t)$ for every $t \in I_j$. Moreover, by (v), $|\bar{x}_j(t) - \tilde{x}(t)| < \epsilon$ for every $t \in I_j$; whence, $\bar{x}_j(t) \in]d_i - 3\epsilon, d_i + 3\epsilon[$. Then, for g nonincreasing on the above interval,

$$(6) \quad g(\bar{x}_j(t)) \leq g(\tilde{x}(t)) \quad \text{for every } t \in I_j.$$

Now, (5) and (6) together give

$$(7) \quad \int_{I_j} g(\bar{x}_j(t)) dt + \int_{I_j} h(\bar{x}_j'(t)) dt \leq \int_{I_j} g(\tilde{x}(t)) dt + \int_{I_j} h^{**}(\tilde{x}'(t)) dt.$$

Let $\bar{x} : [0, T] \rightarrow \mathbb{R}$ be the function whose restriction to each I_j ($j \in \mathbb{N}$) is $\bar{x}_j$.

Then $\bar{x} \in W^{1,p}$ and $\bar{x}(0) = \tilde{x}(0)$, $\bar{x}(T) = \tilde{x}(T)$. Moreover, by (7)

$$\begin{aligned} \inf(\mathbf{P}) &\leq \int_0^T g(\bar{x}(t)) dt + \int_0^T h(\bar{x}'(t)) dt \\ &\leq \int_0^T g(\tilde{x}(t)) dt + \int_0^T h^{**}(\tilde{x}'(t)) dt \\ &= \min(\mathbf{PR}) \leq \inf(\mathbf{P}). \end{aligned}$$

It follows that the above inequalities are in fact equalities: $\bar{x}$ is a solution to (P). $\square$

Remarks. (a) The proof of Theorem 1 holds for a class of maps larger than $\mathcal{D}$. In fact, it is enough to assume that on each interval either $g' = 0$ or g' is of constant sign and that between two intervals having different signs of g' , there is one with $g' = 0$.

(b) The classical example of a variational problem not having a solution is given by

$$h(x') = |x' - 1||x' + 1| \quad \text{and} \quad g(x) = x^2, \quad x(0) = 0 = x(1).$$

By remark (a), by modifying the function $x \mapsto x^2$ in an arbitrarily small neighbourhood of zero as a constant function, one obtains that the modified problem admits solutions for every choice of the boundary data.

(c) The technical tool used, Lemma 1, is very specific to one-dimensionality; in [A-M] a counterexample to the validity of the very same lemma in higher dimension is presented. Still the authors are inclined to believe that a possibly weaker statement, e.g., density of both g and h , should hold in general.

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REFERENCES

- [A-C] M. Amar and A. Cellina, *On passing to the limit for non convex variational problems*, Asymptotic Anal. (to appear).
- [A-M] M. Amar and C. Mariconda, *A non-convex variational problem with constraints*, SIAM J. Control Optim. (to appear).
- [C-C] A. Cellina and G. Colombo, *On a classical problem of the calculus of variations without convexity assumptions*, Ann. Inst. H. Poincaré **7** (1990), 97–106.
- [Ce] L. Cesari, *Optimization-theory and applications*, Springer-Verlag, New York, 1983.
- [E-T] I. Ekeland and R. Temam, *Convex analysis and variational problems*, North-Holland, Amsterdam, 1977.
- [M] P. Marcellini, *Alcune osservazioni sull'esistenza del minimo di integrali del calcolo delle variazioni senza ipotesi di convessità*, Rend. Mat. (2) **13** (1980), 271–281.

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