ON THE METRICAL THEORY OF CONTINUED FRACTIONS

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ABSTRACT. Suppose b_k denotes either $\phi(k)$ or $\phi(p_k)$ ($k = 1, 2, \dots$) where the polynomial ϕ maps $\mathbb{N}$ to $\mathbb{N}$ and p_k denotes the k th rational prime. Suppose $(c_k(x))_{k=1}^\infty$ denotes the sequences of partial quotients of the continued function expansion of the real number x . Then for certain functions $F: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ we show that

$$\lim_{N \rightarrow \infty} F^{-1} \left[\frac{F(c_{b_1}(x)) + \dots + F(c_{b_k}(x))}{N} \right] = F^{-1} \left[\frac{1}{(\log 2)} \int_0^1 \frac{F(c_1(x))}{1+x} dx \right]$$

almost everywhere with respect to Lebesgue measure. This result with $b_k = k$ is classical and due to Ryll-Nardzewski.

In this note we consider the continued fraction expansion of a real number x :

$$x = c_0 + \frac{1}{c_1 + \frac{1}{c_2 + \dots}}$$

Suppose (X, β, μ, T) is a measure-preserving dynamical system. Following [1] we say a strictly increasing sequence of nonnegative integers $(b_n)_{n=1}^\infty$ is L^p -good universal if for each function f in $L^p(X, \beta, \mu)$

$$\lim_{n \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(T^{b_n} x) = l(x),$$

exists μ almost everywhere.

We prove

Theorem 1. Suppose the function $F: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ is continuous, increasing, and that for some $p \geq 1$ we have

$$\int_0^1 \frac{|F(c_1(x))|^p}{x+1} dx < \infty.$$

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For each n in $\mathbb{N}$ and arbitrary nonnegative real numbers $a_1, \dots, a_n$ we define

$$M_{F,n}(a_1, \dots, a_n) = F^{-1} \left[\frac{F(a_1) + \dots + F(a_n)}{n} \right].$$

Assume $(b_n)_{n=1}^\infty$ is any strictly increasing sequence of nonnegative integers satisfying the following two conditions:

- (i) For each irrational number α , if $\langle x \rangle$ denotes the fractional part of x , then $(\langle b_n \alpha \rangle)_{n=1}^\infty$ is uniformly distributed modulo one.
- (ii) The sequence $(b_n)_{n=1}^\infty$ is L^p -good universal.

Then

$$\lim_{n \rightarrow \infty} M_{F,n}(c_{b_1}(x), \dots, c_{b_n}(x)) = F^{-1} \left[\frac{1}{\log 2} \int_0^1 \frac{F(c_1(x))}{1+x} dx \right]$$

almost everywhere with respect to Lebesgue measure.

The special case $b_n = n$ of this result is due to Ryll-Nardzewski [9] who unified earlier work of Khinchine [4]. More general examples of sequences $(b_n)_{n=1}^\infty$ satisfying (i) and (ii) include, if ϕ denotes an arbitrary nonconstant polynomial mapping $\mathbb{N}$ to itself, $b_n = \phi(n)$ and $b_n = \phi(p_n)$. Here p_n denotes the n th rational prime. That both these examples satisfy (i) is well known [11, 8]. That they satisfy (ii) for any $p > 1$ is shown in [2, 6].

We mention some applications of Theorem 1 which follow for appropriate choice of F .

Corollary 2. If the sequence $(b_n)_{n=1}^\infty$ satisfies (i) and (ii) then

$$\lim_{n \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N c_{b_k}(x) = \infty$$

almost everywhere with respect to Lebesgue measure.

Corollary 3. If the sequence $(b_n)_{n=1}^\infty$ satisfies (i) and (ii) then

$$\lim_{N \rightarrow \infty} (c_{b_1}(x) \cdots c_{b_N}(x))^{1/N} = \prod_{k=1}^{\infty} \left(1 + \frac{1}{k^2 + 2k} \right)^{(\log k)/\log 2}$$

almost everywhere with respect to Lebesgue measure.

Corollary 4. Suppose the sequence $(b_n)_{n=1}^\infty$ satisfies (i) and (ii). Let $P_N(x, q)$ ($N = 1, 2, \dots$) denote the number of $c_{b_1}(x), \dots, c_{b_N}(x)$ such that if $c_{b_i}(x) = q$ then

$$\lim_{N \rightarrow \infty} \frac{P_N(x, q)}{N} = \frac{1}{(\log 2)} \log \frac{(q+1)^2}{q(q+2)}$$

almost everywhere with respect to Lebesgue measure.

We remark that Corollary 2 follows by applying Theorem 1 to $F(x) = x$. Strictly speaking this function does not satisfy the hypothesis of Theorem 1, however, setting $F(x) = \min(f(x), M)$ and letting $M \rightarrow \infty$ overcomes this difficulty. See the argument leading to (7) for more details. Corollaries 3 and 4 are straightforward.

All these corollaries are known in the case $b_n = n$. In this special case Corollaries 2 and 3 are due to Khinchine [4] and Corollary 4 is due to Lévi [5].

We say a transformation T of a probability space (X, β, μ) is weakly mixing if for each pair of sets A and B in the sigma-algebra β we have

$$(1) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N |\mu(A \cap T^{-k}B) - \mu(A)\mu(B)| = 0.$$

Let U be the Koopman unitary operator on $L^2(X, \beta, \mu)$ defined pointwise by $f(x) \rightarrow f(Tx)$. We remind the reader that by approximating arbitrary L^2 functions f and g by linear combinations of step functions we see that (1) is equivalent to the formally stronger statement

$$(2) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N |\langle U^k f, g \rangle - \langle f, 1 \rangle \langle g, 1 \rangle| = 0.$$

Here $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $L^2(X, \beta, \mu)$.

To prove Theorem 1 we need some lemmas.

Lemma 5. Suppose $(f_n)_{n=1}^\infty$ is a bounded nonnegative sequence of real numbers. Then the following two statements are equivalent:

- (i) $\sum_{n=1}^N f_n = o(N)$ and
- (ii) $\sum_{n=1}^N f_n^2 = o(N)$.

Proof. In [7] following Koopman and von Neumann it is shown that (i) is equivalent to the statement that there exists $M \subseteq \mathbb{N}$ such that

$$\lim_{N \rightarrow \infty} \frac{\#\{M \cap [1, N]\}}{N} = 1 \quad \text{and} \quad \lim_{\substack{n \rightarrow \infty \\ n \in M}} f_n = 0.$$

The equivalence of (i) and (ii) follows immediately from this. $\square$

We say a sequence $(a_n)_{n=-\infty}^\infty$ is positive definite if for any sequence $\{z_n\}_{n=-\infty}^\infty$ having only finitely many nonzero terms we have

$$\sum_{n,m} a_{n-m} z_n \bar{z}_m \geq 0.$$

We have the Herglotz theorem [3, p. 38].

Lemma 6. A sequence $(a_n)_{n=-\infty}^\infty$ is positive definite if and only if there exists a positive measure μ on $\mathbb{T}$ such that

$$a_n = \int_{\mathbb{T}} e^{-int} d\mu \quad (n \in \mathbb{Z})$$

(i.e., the n th Fourier coefficient of μ denoted by $\hat{\mu}(n)$).

We now have Wiener's criteria [3, p. 42] for the continuity of a measure.

Lemma 7. A measure μ on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ has no atoms if and only if

$$\lim_{n \rightarrow \infty} \frac{1}{2n+1} \sum_{-N}^N |\hat{\mu}(n)|^2 = 0.$$

Lemma 8. Suppose that the measure-preserving dynamical system (X, β, μ, T) is weakly mixing and that the strictly increasing sequence of nonnegative integers

$(\hat{b}_n)_{n=1}^\infty$ satisfies conditions (i) and (ii). Then if $f \in L^p(X, \beta, \mu)$ with $p > 1$ we have $l(x) = \int_X f d\mu$ μ almost everywhere.

Proof. We begin by proving the statement in norm, that is,

$$(3) \quad \lim_{N \rightarrow \infty} \left\| \frac{1}{N} \sum_{k=1}^N f(T^{b_k} x) - \int_X f d\mu \right\| = 0,$$

where $\|f\|$ denotes the $L^2(X, \beta, \mu)$ norm of the function f . By arguing with $f(x) - \int_X f d\mu$ instead of $f(x)$ if necessary, we can assume without loss of generality that $\int_X f d\mu = 0$. Now let U denote the Koopman linear isometry on $L^2(X, \beta, \mu)$. As is immediately verified, letting U^n denote $(U^*)^{|n|}$ if $n < 0$, the sequence $(\langle U^n f, f \rangle)_{n=-\infty}^\infty$ is positive definite; hence, there is a measure ω_f on $\mathbb{T}$, dependent on f such that

$$\langle U^n f, f \rangle = \int_{\mathbb{T}} e^{-int} d\omega_f(t) \quad (n \in \mathbb{Z}).$$

Some basic properties of U can be found in [10, p. 25]. This means that

$$\begin{aligned} \left\| \frac{1}{N+1} \sum_{k=0}^N f(T^{b_k} x) \right\|^2 &= \frac{1}{(N+1)^2} \sum_{0 \leq k_1, k_2 \leq N} \langle U^{b_{k_2} - b_{k_1}} f, f \rangle \\ (4) \quad &= \frac{1}{(N+1)^2} \sum_{0 \leq k_1, k_2 \leq N} \int_0^1 e^{2\pi i(b_{k_1} - b_{k_2})\theta} d\omega_f(\theta) \\ &= \int_0^1 \left| \frac{1}{N+1} \sum_{k=0}^N e^{2\pi i b_k \theta} \right|^2 d\omega_f(\theta). \end{aligned}$$

By Lemma 7 the measure ω_f is nonatomic if we can show that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N |\hat{\omega}_f(n)|^2 = 0.$$

Because $|\hat{\omega}_f(n)| \leq \|\omega_f\|$ for each $n \in \mathbb{Z}$, by Lemma 5, it is sufficient to show

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N |\hat{\omega}_f(n)| = 0.$$

Now note that the weak-mixing of T in the form (2) and the fact that $\int_X f d\mu = 0$ give

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N |\hat{\omega}_f(n)| = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N |\langle U^k f, f \rangle| = 0.$$

This means ω_f is nonatomic. It now follows that (3) holds using (4) and (i), that is, the fact that $(\langle b_k \theta \rangle)_{k=1}^\infty$ is uniformly distributed modulo one for each irrational θ .

To prove the almost everywhere convergence we argue as follows. Let $(N_t)_{t=1}^\infty$ denote a strictly increasing sequence of integers such that

$$\left\| \frac{1}{N_t} \sum_{k=1}^{N_t} f(T^{b_k} x) - \int_X f d\mu \right\| < \frac{1}{t}.$$

Thus

$$\sum_{t=1}^{\infty} \left\| \frac{1}{N_t} \sum_{k=1}^{N_t} f(T^{b_k} x) - \int_X f d\mu \right\|^2 < \infty.$$

Hence rearranging using Fatou's lemma we have

$$\int_X \left(\sum_{t=1}^{\infty} \left\| \frac{1}{N_t} \sum_{k=1}^{N_t} f(T^{b_k} x) - \int_X f d\mu \right\|^2 \right) d\mu < \infty.$$

Thus using the fact that if $\sum a_n < \infty$ with $a_n \geq 0$ then $a_n = o(1)$, we have

$$(5) \quad \lim_{t \rightarrow \infty} \frac{1}{N_t} \sum_{k=1}^{N_t} f(T^{b_k} x) = \int_X f d\mu \quad \text{a.e.}$$

By (ii), that is, the L^p -good universality property, the limit exists and so it must be $\int_X f d\mu$ as required by (5). $\square$

We now complete the proof of Theorem 1. Let T denote the map of $(0, 1)$ defined by $Tx = \langle 1/x \rangle$. We note that the partial quotients $(c_k(x))_{k=1}^{\infty}$ of x satisfy $c_k(x) = c_1(T^{k-1}x)$. Gauss observed that the map T preserves the measure

$$(6) \quad \mu(A) = \frac{1}{\log 2} \int_A \frac{dx}{1+x}.$$

This measure is equivalent to Lebesgue measure on $[0, 1]$ (in the sense that it has the same null sets). In addition the transformation T is weak mixing on $[0, 1]$ with respect to the measure μ in (6). If $\int_0^1 F(c_1(x)) dx / (1+x)$ is finite then Theorem 1 follows immediately from Lemma 8. If $\int_0^1 F(c_1(x)) dx / (1+x)$ is not finite, we argue as follows. Let

$$f_M(x) = \begin{cases} F(c_1(x)) & \text{if } |F(c_1(x))| \leq M, \\ M & \text{if } |F(c_1(x))| > M. \end{cases}$$

This means that for each $M \geq |F(1)|$ and for almost all x

$$(7) \quad \begin{aligned} \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N F(c_{b_k}(x)) &\geq \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f_M(T^{b_k-1}x) \\ &= \frac{1}{(\log 2)} \int_0^1 \frac{f_M(x)}{1+x} dx. \end{aligned}$$

Now letting M tend to infinity completes the proof.

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