ON A DEFINITE INTEGRAL OF A HYPERGEOMETRIC FUNCTION

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ABSTRACT. The connections between the elliptic functions and the hypergeometric series ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)$ is well known and classical. In this note, we investigate its relation with ${}_2F_1(\frac{1}{4}, \frac{3}{4}; 1; x)$. We find that it is less ideal than the classical case and discuss the flaws.

1. INTRODUCTION

Recently Berndt, Bhargava, and Garvan [3] provided the first proof of the following remarkable statement of Ramanujan.

Theorem A. For $0 \leq q < 1$, let $a = \vartheta_3(q)\vartheta_3(q^3) + \vartheta_2(q)\vartheta_2(q^3)$, $c = \frac{1}{2}a(q^{1/3}) - \frac{1}{2}a(q)$, and $x = c^3/a^3$; and for $0 \leq \varphi \leq \pi/2$ define

$$(1) \quad za = \int_0^\varphi {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; x \sin^2 t\right) dt.$$

Then, for $0 \leq z \leq \pi/2$

$$(2) \quad \varphi = z + 3 \sum_{n=1}^{\infty} \frac{q^n \sin 2nz}{n(1 + q^n + q^{2n})}.$$

Despite its seemingly mysterious appearance, Theorem A actually has a familiar analogue:

Theorem B. Let $a = \vartheta_3^2(q)$, $c = \vartheta_2^2(q)$, and $k^2 = c^2/a^2$. Define

$$(3) \quad za = \int_0^\varphi {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; k^2 \sin^2 t\right) dt.$$

Then

$$(4) \quad \varphi = z + 2 \sum_{n=1}^{\infty} \frac{q^n \sin 2nz}{n(1 + q^{2n})}.$$

The proof of Theorem B goes as follows: We recall [4, p. 101]

$$(5) \quad {}_2F_1\left(\frac{1}{2}a + \frac{1}{2}, \frac{1}{2} - \frac{1}{2}a; \frac{1}{2}; \sin^2 z\right) = \frac{\cos az}{\cos z}.$$

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Let $a = 0$ and $x = \sin^2 z$. Then (5) becomes

$$(6) \quad {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; x\right) = \frac{1}{\sqrt{1-x}}, \quad |x| < 1.$$

From (6), we see that by letting $x = k^2 \sin^2 \theta$ and $t = \sin \theta$, we can write (3) as

$$\begin{aligned} z\vartheta_3^2 &= \int_0^\varphi {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; k^2 \sin^2 \theta\right) d\theta \\ &= \int_0^\varphi (1 - k^2 \sin^2 \theta)^{-1/2} d\theta = \int_0^{\sin \varphi} \frac{dt}{\sqrt{(1-t^2)(1-k^2 t^2)}}. \end{aligned}$$

Thus, from the definition of $sn(z, k)$ (see [5, p. 492]), we see that

$$(7) \quad \sin \varphi = sn(\vartheta_3^2 z).$$

To solve φ in terms of z , we differentiate (7) with respect to z . Then (see [5, p. 511])

$$\begin{aligned} (8) \quad \frac{d\varphi}{dz} &= \vartheta_3^2 \frac{cn(\vartheta_3^2 z) dn(\vartheta_3^2 z)}{\cos \varphi} = \vartheta_3^2 dn(\vartheta_3^2 z) \\ &= 1 + 4 \sum_{n=1}^{\infty} \frac{q^n \sin 2nz}{1 + q^{2n}}. \end{aligned}$$

The identity (4) now follows immediately by integrating (8). This completes the proof of Theorem B.

It is worthwhile to point out that, when we let $\varphi = \pi/2$ in (1) and (3), they yield the following interesting identities:

(i) Let $a = \vartheta_3^2$, $b = \vartheta_4^2$, and $c = \vartheta_2^2$. Then [5, p. 499]

$$a^2 = b^2 + c^2 \quad \text{and} \quad a = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{c^2}{a^2}\right).$$

(ii) Let a and c be as in Theorem A and $b = \frac{3}{2}a(q^3) - \frac{1}{2}a(q)$.

Then

$$a^3 = b^3 + c^3 \quad \text{and} \quad a = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \frac{c^3}{a^3}\right).$$

The proof can be found in [2].

We note that Ramanujan also discovered two other similar identities of this sort. Namely:

(iii) Let $a = \sqrt{(\vartheta_3^4(q) + \vartheta_4^4(q))/2}$, $b = \vartheta_3(q)\vartheta_4(q)$, and $c = \vartheta_2^2(q)/\sqrt{2}$.

Then

$$a^4 = b^4 + c^4 \quad \text{and} \quad a = {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \frac{c^4}{a^4}\right).$$

(iv) Let $a = E_4^{1/4}$, $b = (\frac{1}{2}(E_4^{3/2} + E_6))^{1/6}$, and $c = (\frac{1}{2}(E_4^{3/2} - E_6))^{1/6}$, where E_4 and E_6 are the familiar Eisenstein series. Then

$$a^6 = b^6 + c^6 \quad \text{and} \quad a = {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; \frac{c^6}{a^6}\right).$$

The proof of (iii) follows quite easily from applying the quadratic transformations of the hypergeometric series. To see this, we appeal to the quadratic transformation [4, p. 112, (26) and (28)]

$${}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \left(\frac{z}{2-z}\right)^2\right) = \sqrt{\frac{2-z}{2}} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; z\right).$$

Then second identity of (iii) follows immediately from choosing $z = \vartheta_2^4/\vartheta_3^4$. The proof of (iv) involves cubic transformations of the hypergeometric series and is thus much less straightforward. Naturally, in view of the above discussion, we are lead to the following analogue of Theorems A and B.

Theorem C. *Let a and c be defined as in (iii). Define*

$$(8) \quad za = \int_0^\varphi {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; \alpha \sin^2 t\right) dt, \quad \alpha = \frac{c^4}{a^4}.$$

Then

$$(9) \quad \cos \varphi = cn(u, k) dn(u, k),$$

where $u = \vartheta_3^2(q^2)z$ and $k = \vartheta_2^2(q^2)/\vartheta_3^2(q^2)$.

One feature of (9) is quite interesting. Namely, the differentiation of (7) “almost” yields (9). So in view of (2) and (4), it is natural to ask if one can solve φ in terms of z and obtain an analogue similar to (2) and (4). This, however, seems difficult, and we will discuss this question in the end of this paper.

It is important to observe here that in (8) the quantities a and α are defined in terms of $\vartheta_i(q)$, $i = 2, 3, 4$, whereas $\cos \varphi$ in (9) is an elliptic function of modulus k which is defined in terms of $\vartheta_i(q^2)$. One therefore anticipates the involvement of quadratic transformations of theta functions in the proof of Theorem C.

In view of (ii) and (iii), one can regard Theorems A and C as, respectively, the cubic and quartic analogues of Theorem B.

We assume the readers are reasonably familiar with the properties and the basic notation of theta functions and the elliptic functions of Jacobi and Weierstrass.

2. PROOF OF THEOREM C

In (5), we choose $a = \frac{1}{2}$ and set $t = \sin^2 z$. Then

$${}_2F_1\left(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; t\right) = \frac{1}{\sqrt{2}} \left(\frac{1 + (1-t)^{1/2}}{1-t}\right)^{1/2}.$$

Hence (8) becomes

$$(10) \quad \begin{aligned} za &= \int_0^\varphi {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; \alpha \sin^2 \theta\right) d\theta \\ &= \frac{1}{\sqrt{2}} \int_0^\varphi \left(\frac{1 + (1 - \alpha \sin^2 \theta)^{1/2}}{1 - \alpha \sin^2 \theta}\right)^{1/2} d\theta \\ &= \frac{1}{\sqrt{2}} \int_0^{\sin \varphi} \left(\frac{1 + (1 - \alpha t^2)^{1/2}}{(1-t^2)(1-\alpha t^2)}\right)^{1/2} dt. \end{aligned}$$

So, to establish Theorem C, we are led to the consideration of the integral

$$(11) \quad \sqrt{2}az = \int_0^y \left(\frac{1 + (1 - \alpha t^2)^{1/2}}{(1 - t^2)(1 - \alpha t^2)} \right)^{1/2} dt, \quad \text{where } y = \sin \varphi.$$

First we note that making the substitution $x = \sqrt{1 - \alpha t^2}$, the above integral becomes

$$(12) \quad -\sqrt{2}az = \int_1^w ((1 - x)(x^2 - (1 - \alpha)))^{-1/2} dx, \quad \text{where } w = \sqrt{1 - \alpha y^2}.$$

The polynomial in the integrand of (12) is cubic, which immediately suggests its connection with the Weierstrassian elliptic function. Indeed, following the standard procedures described in [5, p. 453], we see that, through several changes of variables, the integral (12) is converted to the standard form

$$(13) \quad -\sqrt{2}az = \int_s^\infty \left\{ 4t^3 - \left(\frac{1}{3} - \frac{\alpha}{4} \right) t - \left(\frac{1}{27} - \frac{\alpha}{24} \right) \right\}^{-1/2} dt,$$

where $s = \alpha/4(1 - w) - 1/6$. From this, we immediately see that, as a function of z , w can be expressed as

$$(14) \quad w(z) = 1 - \frac{\alpha}{4} \left\{ \wp(\sqrt{2}az; g_2, g_3) + \frac{1}{6} \right\}^{-1},$$

where

$$g_2 = \frac{1}{3} - \frac{\alpha}{4} \quad \text{and} \quad g_3 = \frac{1}{27} - \frac{\alpha}{24}.$$

One can easily verify that the three distinct roots of the cubic polynomial in (13) are

$$e_1 = \frac{1 + \sqrt{1 - \alpha}}{4} - \frac{1}{6}, \quad e_2 = \frac{1 - \sqrt{1 - \alpha}}{4} - \frac{1}{6}, \quad e_3 = -\frac{1}{6}.$$

Expressing $\wp(\sqrt{2}az; g_2, g_3)$ in terms of Jacobian elliptic functions (see [5, p. 505]), we see that (14) becomes

$$(15) \quad w(z) = 1 - \frac{\alpha}{4(e_1 - e_3)} \operatorname{sn}^2(\sqrt{2}a(e_1 - e_3)^{1/2}z, k),$$

where the modulus k is given by the equation

$$(16) \quad k^2 = \frac{e_2 - e_3}{e_1 - e_3} = \frac{1 - \sqrt{1 - \alpha}}{1 + \sqrt{1 - \alpha}}.$$

To compute $e_1 - e_3$, $e_2 - e_3$, and k more explicitly, we need

Lemma 1 [1, p. 35].

$$\vartheta_2^2(q) = 2\vartheta_2(q^2)\vartheta_3(q^2), \quad \vartheta_3^2(q) = \vartheta_3^2(q^2) + \vartheta_2^2(q^2), \quad \vartheta_4^2(q) = \vartheta_3^2(q^2) - \vartheta_2^2(q^2).$$

From Lemma 1, we have

$$(17) \quad \begin{aligned} \vartheta_3^4(q) + \vartheta_4^4(q) &= 2(\vartheta_3^4(q^2) + \vartheta_2^4(q^2)), \\ (\vartheta_3^4(q) + \vartheta_4^4(q))^2 &= \vartheta_2^8(q) + 4\vartheta_3^4(q)\vartheta_4^4(q). \end{aligned}$$

Incidentally, the second identity in (17) is exactly the statement that $a^4 = b^4 + c^4$. Therefore, from (17)

$$(18) \quad \alpha = \frac{c^4}{a^4} = \frac{\vartheta_2^8(q)}{(\vartheta_3^4(q) + \vartheta_4^4(q))^2} = \frac{4\vartheta_2^4(q^2)\vartheta_3^4(q^2)}{(\vartheta_3^4(q^2) + \vartheta_2^4(q^2))^2} = \frac{4k^2}{(1+k^2)^2}$$

and

$$(19) \quad \sqrt{1-\alpha} = \frac{1-k^2}{1+k^2}, \quad \text{where } k = \frac{\vartheta_2^2(q^2)}{\vartheta_3^2(q^2)}.$$

Hence, from (19),

$$(20) \quad \sqrt{e_1 - e_3} = \vartheta_3^2(q^2)/\sqrt{2}a.$$

From (15) and (20),

$$(21) \quad \sqrt{1-\alpha \sin^2 \varphi} = w(z) = 1 - \frac{2k^2}{1+k^2} \operatorname{sn}^2(u, k) \quad (u = \vartheta_3^2(q^2)z).$$

Squaring both sides of (21), we obtain

$$\sin^2 \varphi = \sin^2 u(1+k^2-k^2 \operatorname{sn}^2 u) = \operatorname{sn}^2 u(k^2 + dn^2 u).$$

Thus

$$\begin{aligned} \cos^2 \varphi &= 1 - \sin^2 \varphi = 1 - k^2 \operatorname{sn}^2 u - \operatorname{sn}^2 u dn^2 u \\ &= dn^2 u(1 - \operatorname{sn}^2 u) = cn^2 u dn^2 u. \end{aligned}$$

Therefore, $\cos \varphi = cnudnu$. This establishes Theorem C.

We now discuss the relation between φ and z . To do so, we differentiate (9) with respect to z . Then

$$(22) \quad \frac{d\varphi}{dz} = \vartheta_3^2(q^2) \frac{(k^2 cn^2 u + dn^2 u) sn u}{\sin \varphi} = \vartheta_3^2(q^2) \frac{(k^2 cn^2 u + dn^2 u)}{\sqrt{k^2 + dn^2 u}}.$$

Clearly, $d\varphi/dz$ is even, of period π , and analytic for z real. Hence the Fourier series expansion of $d\varphi/dz$ is of the form

$$\frac{d\varphi}{dz} = \sum_{n=0}^{\infty} a_n \cos 2nz.$$

Integrating both sides and using the fact that $\varphi = \pi/2$ when $z = \pi/2$, we see that

$$\varphi = z + \frac{1}{2} \sum_{n=0}^{\infty} \frac{a_n}{n} \sin 2nz.$$

However, because of the presence of the radical in (22), $d\varphi/dz$ is not an elliptic function; therefore, the coefficients a_n seem difficult to evaluate. From this prospective, the “corresponding theory” of elliptic functions based on ${}_2F_1(\frac{1}{4}, \frac{3}{4}; 1; x)$ seems less satisfactory in this respect.

From a different angle, one can almost anticipate this difficulty of expressing φ in terms for z . This is because when we differentiate (1) and (3) with respect to z and set $z = 0$, we obtain, respectively,

$$\vartheta_3(q)\vartheta_3(q^3) + \vartheta_2(q)\vartheta_2(q^3) = \left. \frac{d\varphi}{dz} \right|_{z=0} = 1 + 6 \sum_{n=1}^{\infty} \frac{q^n}{1+q^n+q^{2n}}$$

and

$$\vartheta_3^2(q) = 1 + 4 \sum_{n=1}^{\infty} \frac{q^n}{1 + q^{2n}};$$

it is therefore reasonable to expect that $d\varphi/dz$ for a general value z will possess a similar feature and this is indeed so. But the differentiation of (8) yields

$$\left. \frac{d\varphi}{dz} \right|_{z=0} = \sqrt{\frac{\vartheta_4^4(q) + \vartheta_3^4(q)}{2}},$$

which is not known to be expressible in terms of simple Lambert series.

We now end our discussion by observing that

$$a^2 = \frac{\vartheta_4^4(q) + \vartheta_3^4(q)}{2} = {}_2F_1^2\left(\frac{1}{4}, \frac{3}{4}; 1; \frac{c^4}{a^4}\right)$$

does have simple Lambert series expansion. We are lead to the consideration of the quantity ${}_2F_1^2(\frac{1}{4}, \frac{3}{4}; 1; x)$, and it is interesting to note that this ties naturally to the identity (see [1, Theorem 5.7(ii)])

(23)

$$b^2 = {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; -\frac{c^4}{b^4}\right) = \frac{2}{\pi} \int_0^{\pi/2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; -\frac{c^4}{b^4} \sin^2 t\right) dt.$$

To see this, we recall [4, p. 105]

$${}_2F_1(\alpha, \beta; \gamma; z) = (1-z)^{-\beta} {}_2F_1(\gamma-\alpha, \beta; \gamma; z/(1-z))$$

and Clausen's product formula [1, p. 188]

$${}_2F_1^2(\alpha, \beta; \alpha + \beta + \frac{1}{2}; z) = {}_3F_2(2\alpha, \alpha + \beta, 2\beta; \alpha + \beta + \frac{1}{2}, 2\alpha + 2\beta; z).$$

Choosing $\alpha = \beta = \frac{1}{4}$, $\gamma = 1$, and $z = -c^4/b^4$, we obtain the first part of the identity (23). The second part of the identity is an easy consequence of the fact that

$$\int_0^{\pi/2} \sin^{2n} t dt = \frac{\pi}{2} \frac{(\frac{1}{2})_n}{n!}.$$

In view of the central theme of this paper, it will be interesting to consider the integral

$$zb^2 = \int_0^\varphi {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; -\frac{c^4}{b^4} \sin^2 t\right) dt$$

and ask whether a relation similar to (2) and (4) can be obtained.

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