APPROXIMATION BY SEMI-FREDHOLM OPERATORS

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ABSTRACT. We present a simple approach for calculating some known distances from a bounded operator defined on a separable Hilbert space to certain sets related to semi-Fredholm operators.

1. INTRODUCTION

There are several semigroups which appear naturally in connection with the Banach algebra $B(X)$ of all (bounded linear) operators acting from a Banach space X into itself. We have the injective operators

$$\text{Inj}(X) = \{T \in B(X) : T \text{ is one-to-one, and } R(T) \text{ is closed}\},$$

the surjective operators

$$\text{Sur}(X) = \{T \in B(X) : R(T) = X\},$$

the upper semi-Fredholm operators

$$\Phi_+(X) = \{T \in B(X) : \dim N(T) < \infty \text{ and } R(T) \text{ is closed}\},$$

and the lower semi-Fredholm operators

$$\Phi_-(X) = \{T \in B(X) : \dim X/R(T) < \infty\};$$

here $N(T)$ denotes the null space of T and $R(T)$ is its range. It is well known that these semigroups are open [5, Theorem 2.5.6 and 4.2.1], and we clearly have $\text{Inj}(X) \subset \Phi_+(X)$ and $\text{Sur}(X) \subset \Phi_-(X)$. Although the contentions are generally proper, Lemma 2.1 shows that $\text{Inj}(X) \cup \text{Sur}(X)$ is dense in $\Phi_+(X) \cup \Phi_-(X)$.

Recall that the index of a (upper or lower) semi-Fredholm operator T is $\text{ind } T = n(T) - d(T)$, where $n(T) = \dim N(T)$ and $d(T) = \dim X/R(T)$. Let Z^* consist of all the integers together with $-\infty$ and ∞ . For $m \in Z^*$, we define $\Phi_m(X)$ as the set of semi-Fredholm operators $T \in B(X)$ such that $\text{ind } T = m$.

If H is a complex separable Hilbert space, Bouldin [2–4], Zemánek [8], Apostol et al. [1], Izumino and Kato [6], and Wu [7] have calculated the distance

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of $T \in B(H)$ to several subsets related to semi-Fredholm operators on H . We develop an alternative approach for obtaining some of their results.

2. APPROXIMATION BY SEMI-FREDHOLM OPERATORS

Let us consider

$$F_m(X) = \begin{cases} \Phi_m(X) \cap \text{Sur}(X), & m \geq 0, \\ \Phi_m(X) \cap \text{Inj}(X), & m \leq 0. \end{cases}$$

The following result is probably well known, but we have not found it stated explicitly. We present it here for the sake of completeness.

Lemma 2.1. $F_m(X)$ is dense in $\Phi_m(X)$, $m \in \mathbb{Z}^*$.

Proof. First we will assume that $m \leq 0$. Take $T \in \Phi_m(X)$. Since $m \leq 0$, note that $n = n(T) < \infty$. Thus we can express $X = N(T) \oplus V$, for some closed subspace $V \subset X$. Next, by using $d(T) \geq n$, we can find a vector space $W \subset X$ such that $\dim W = n$ and $W \cap R(T) = 0$. Take S to be any linear isomorphism from $N(T)$ onto W . Let us define $T_k: X \rightarrow X$ by

$$T_k(u + v) = \frac{1}{k}Su + Tv, \quad u \in N(T), \quad v \in V.$$

Since the projections onto $N(T)$ and V are continuous, as well as S , it follows that $T_k \in B(X)$. It is clear that $T_k \in \text{Inj}(X) \cap \Phi_m(X)$. Noticing that $T_k \rightarrow T$, we obtain the desired conclusion.

Let us now consider $m \geq 0$. Then we have $d = d(T) < \infty$, so we can express $X = N \oplus V$, where $N \subset N(T)$ and $\dim N = d$. The rest of the proof follows along similar lines as above, and we will omit it.

Remark 2.1. The proof of Lemma 2.1 shows that if $T \in \Phi_m(X)$, then there is a compact operator $K \in B(X)$ such that $T + K \in F_m(X)$.

It will now be convenient to introduce some notation. Hereafter H is always a fixed complex separable Hilbert space. If $A \subset H$, then $A^c \equiv H \setminus A$; for $T \in B(H)$, T^* is its adjoint operator and $|T|$ is the square root of T^*T . The group of invertible operators on H will be indicated by $G(H)$; moreover, $\Phi_+ \equiv \Phi_+(H)$, $\Phi_- \equiv \Phi_-(H)$, $\Phi_m \equiv \Phi_m(H)$.

Let us denote the semigroup of Fredholm operators on H by Φ ; notice that $\Phi = \Phi_+ \cap \Phi_-$. The essential spectrum of $T \in B(H)$ is then given by $\sigma_e(T) = \{\lambda: T - \lambda I \notin \Phi\}$, and the essential minimum modulus is

$$m_e(T) = \inf\{\lambda: \lambda \in \sigma_e(|T|)\}.$$

Our approach for calculating the distance $\text{dist}(T, A)$ of $T \in B(H)$ to certain subsets A of $B(H)$ relies on the set

$$N(H) = \{T \in B(H): m_e(T) = m_e(T^*) = 0\},$$

together with the following result (part(i) is due to Bouldin [2, Theorem 3], and part (ii) is due to Zemánek [8]).

Theorem 2.1. (i) If $T \in N(H)$, then $\text{dist}(T, G(H)) = 0$.

(ii) $m_e(T) = \text{dist}(T, \Phi_+^c)$, $m_e(T^*) = \text{dist}(T, \Phi_-^c)$.

Remark 2.2. $N(H)$ is a closed subgroup of $B(H)$.

Proof. Let $T \in B(H)$. The following properties are well known [2]:

$$\begin{aligned} m_e(T) > 0 & \text{ if and only if } T \in \Phi_+, \\ T \in \Phi_- & \text{ if and only if } T^* \in \Phi_+. \end{aligned}$$

This implies that $N(H) = (\Phi_+ \cup \Phi_-)^c$, so $N(H)$ is clearly closed.

Assume, $S, T \in N(H)$. Since

$$(1) \quad TS \in \Phi_- \text{ implies } T \in \Phi_-$$

and $TS \in \Phi_+$ implies $S \in \Phi_+$ [5, Corollary 1.3.3], we have $TS \in N(H)$.

In our next result $\text{bdy } A$ denotes the boundary of $A \subset B(H)$.

Theorem 2.2. For each $m \in Z^*$, we have $N(H) = \text{bdy } \Phi_m$.

Proof. By the continuity of the index on $\Phi_+ \cup \Phi_-$, it follows that $\text{bdy } \Phi_m \subset N(H)$.

Next, let us assume $m \in Z \cup \{-\infty\}$. Fix $R \in \Phi_m \subset \Phi_+$. Take P and Q to be the orthogonal projections onto the null space and the range of R , respectively, and let U be the inverse operator of $T: (I - P)H \rightarrow R(H)$. Then $UQR = I - P$, so, with $S = UQ$, we have $S \in B(H)$ and

$$(2) \quad SR = I + K,$$

where $K \in B(H)$ is a compact operator. Take $T \in N(H)$, and note that $T \notin \Phi_m$. By (1), $TS \notin \Phi_-$. Suppose that $TS \in \Phi_+$. Then, using (2), we would have $T \in \Phi_+$, which is contradictory.

The above shows that $TS \in N(H)$. Thus, by (i) of Theorem 2.1, there is a sequence $\{L_k\} \subset G(H)$ such that $L_k \rightarrow TS$. Hence, applying (2), we have $L_k R \rightarrow T + TK$. Since $\text{ind}(L_k R - TK) = m$, it follows that $T \in \text{bdy } \Phi_m$. This shows that $N(H) = \text{bdy } \Phi_m$.

Finally let us treat the case $m = \infty$. Take $T \in N(H)$. Then $T^* \in N(H)$, so, applying what we have just proved, there is a sequence $\{S_k\} \subset \Phi_{-m}$ such that $S_k \rightarrow T^*$. Hence $S_k^* \rightarrow T$, and the conclusion follows.

Corollary 2.1. Let $m \in Z^*$. If $T \notin \Phi_m$, then

$$\text{dist}(T, \Phi_m) = \max\{m_e(T), m_e(T^*)\}.$$

Proof. First, we notice that $m_e(T) = m_e(T^*)$ when both of these quantities are positive [2, Theorem 2]. Thus

$$\begin{aligned} \max\{m_e(T), m_e(T^*)\} &= m_e(T) \quad \text{if } T \in \Phi_+, \\ \max\{m_e(T), m_e(T^*)\} &= m_e(T^*) \quad \text{if } T \in \Phi_-. \end{aligned}$$

Take $d \equiv \text{dist}(T, \Phi_m)$. We will consider several cases. If $T \notin \Phi_+ \cup \Phi_-$, the conclusion follows readily from Theorem 2.2.

Now assume $T \in \Phi_+$, and let $0 < \rho < m_e(T)$. By Theorem 2.1(ii) the closed ball V consisting of those $R \in B(H)$ such that $\|R - T\| < \rho$ is contained in Φ_+ . Since V is connected, by the continuity of the index, $\text{ind } R = \text{ind } T$ when $R \in V$. This implies

$$(3) \quad m_e(T) \leq d.$$

Let $\varepsilon > 0$. Again applying Theorem 2.1, we can obtain $S \notin \Phi_+$ satisfying $\|T - S\| \leq m_e(T) + \varepsilon$. Now note that, because of the continuity of the index on $\Phi_+ \cup \Phi_-$, the curve $\alpha(t) = (1 - t)T + tS$, $0 \leq t \leq 1$, cannot be contained

in $\Phi_+ \cup \Phi_-$. Thus we can find $0 \leq t \leq 1$ such that $\alpha(t) \in N(H)$. Since $\|T - \alpha(t)\| \leq m_e(T) + \varepsilon$, we have $d \leq m_e(T) + \varepsilon$. Letting $\varepsilon \rightarrow 0$ and using (3), the conclusion follows.

Finally, the case $T \in \Phi_-$ can be proved analogously.

Remark 2.3. Corollary 2.1 was established independently by Apostol et al. [1, Theorem 12.2] and Izumino and Kato [6, Theorem 4.1]; it is also discussed by Bouldin [3].

Corollary 2.2. *If $n \leq 0$, then*

$$\text{dist}(T, F_n) = \begin{cases} \max\{m_e(T), m_e(T^*)\} & \text{if } \text{ind } T \neq n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It follows from Lemma 2.1 that $\text{dist}(T, F_n(H)) = \text{dist}(T, \Phi_n)$. The conclusion is now obtained by using Corollary 2.1.

Remark 2.4. Corollary 2.2 was established by Wu in [7].

Let us denote by ν the function assigning $n(T)$ to each $T \in B(H)$.

Corollary 2.3. *The function ν is continuous at $T \in B(H)$ if and only if $T \in \Phi_+ \cup \Phi_-$ and $n(T) = 0$ or $d(T) = 0$.*

Proof. Let $A \equiv \{T \in \Phi_+ \cup \Phi_- : n(T) = 0, \text{ or } d(T) = 0\}$. If $T \notin A$, it follows from Lemma 2.1 and Theorem 2.2 that ν is discontinuous at T . Now assume $T \in A$. Take $\rho > 0$ such that $\|T - S\| \leq \rho$ implies $S \in (\Phi_+ \cup \Phi_-)$, $n(S) \leq n(T)$, and $d(T) \leq d(S)$ [5, Theorem 4.2.1]. If $n(T) = 0$ and $\|T - S\| \leq \rho$, we clearly have $n(T) = n(S)$. If $d(T) = 0$ and $\|T - S\| \leq \rho$, then $n(T) = n(S)$ because $\text{ind}(T) = \text{ind}(S)$. This proves our assertion.

NOTE ADDED IN PROOF

After this work was sent for publication, a paper of M. Mbekhta appeared which contained results closely related to ours but using different methods (*Sur la structure des composantes connexes semi-Fredholm de $B(H)$* , Proc. Amer. Math. Soc. **116** (1992), 521–524).

REFERENCES

1. C. Apostol, L. A. Faikow, D. A. Herrero and D. Voiculescu, *Approximation by Hilbert space operators*, Vol. II, Pitman, Boston, MA, 1984.
2. R. Bouldin, *The essential minimum modulus*, Indiana Univ. Math. J. **30** (1981), 513–517.
3. —, *The distance to operators with a fixed index*, Acta Sci. Math. (Szeged) **54** (1990), 139–143.
4. —, *Approximation by operators with fixed nullity*, Proc. Amer. Math. Soc. **103** (1988), 141–144.
5. S. R. Caradus, W. E. Pfaffenberger, and B. Yood, *Calkin algebras and algebras of operators on Banach spaces*, Marcel Dekker, New York, 1974.
6. S. Izumino and Y. Kato, *The closure of invertible operator on a Hilbert space*, Acta Sci. Math. (Szeged) **49** (1985), 321–327.
7. P. Y. Wu, *Approximation by invertible and noninvertible operators*, J. Approx. Theory **56** (1989), 267–276.
8. J. Zemánek, *Geometric interpretations of the essential minimum modulus in Invariant subspaces and other topics*, Oper. Theory, Adv. Appl., vol. 6, Birkhäuser Verlag, Basel, 1982, pp. 225–227.