ON QUASI-CONTINUOUS RINGS

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ABSTRACT. A well-known result of Utumi asserts that a two-sided continuous two-sided artinian ring is quasi-Frobenius. In this paper we extend Utumi's result to quasi-continuous rings.

INTRODUCTION AND DEFINITIONS

A well-known result of Utumi [7] asserts that a two-sided continuous two-sided artinian ring is quasi-Frobenius. In [4] this result was extended to two-sided continuous rings with ACC on essential left and right ideals. In [3], motivated by a result of Carl Faith on self-injective rings, it was shown that a two-sided continuous ring with ACC on left annihilators is quasi-Frobenius. There are examples of two-sided artinian one-sided continuous rings which are not quasi-Frobenius (see [4]). In this paper we extend Utumi's result to quasi-continuous rings.

Throughout this paper all rings considered are associative with identity and all modules are unitary R -modules. We write $J(M)$, $Z(M)$, $\text{Soc}(M)$, and $E(M)$ for the Jacobson radical, the singular submodule, the socle, and the injective hull of ${}_R M$, respectively. For any subset X of R , $l_R(X)$ represents the left annihilator of X in R .

Consider the following conditions on a module ${}_R M$:

- (C1) Every submodule of M is essential in a summand of M .
- (C2) Every submodule isomorphic to a summand of M is itself a summand.
- (C3) If M_1 and M_2 are summands of M with $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a summand of M .

M is called *continuous* if it satisfies conditions (C1) and (C2), *quasi-continuous* if it satisfies (C1) and (C3), and a *CS-module* if it satisfies condition (C1) only.

It is easy to see that (C2) implies (C3), but the converse is not true in general. Thus, every continuous module is quasi-continuous. The ring of integers $\mathbb{Z}$ is

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an example of a commutative, noetherian, quasi-continuous ring which is not continuous (and hence not quasi-Frobenius). For a full account of the subject of continuous and quasi-continuous modules we refer the reader to [5].

THE RESULT

Theorem 1. *A left artinian two-sided quasi-continuous ring is quasi-Frobenius.*

Before we begin the proof we need the following lemmas.

Lemma 1. $\bigoplus_{i=1}^n M_i$ is quasi-continuous if and only if each M_i is quasi-continuous and M_j -injective for all $j \neq i$.

Proof. See [5, Corollary 2.14].

Lemma 2. Let M_1 and M_2 be summands of a quasi-continuous module M . If $E(M_1) \cong E(M_2)$, then $M_1 \cong M_2$.

Proof. See [5, Theorem 2.31].

Lemma 3. Let R be a left quasi-continuous ring. Then R is left continuous if and only if $Z({}_R R) = J(R)$ and $R/J(R)$ is regular.

Proof. See [5, Proposition 3.15].

Lemma 4. Let R be a two-sided continuous ring with ACC on left annihilators. Then R is a quasi-Frobenius ring.

Proof. See [3, Theorem 1].

Lemma 5. Let R be a semiperfect ring. Then $Z({}_R R) \subseteq J(R)$ and $Z(R_R) \subseteq J(R)$.

Proof. Since R is semiperfect, $R = Re \oplus R(1 - e)$ such that $Re \subseteq Z({}_R R)$, $Z({}_R R) \cap R(1 - e)$ is small in R , and $e^2 = e \in R$. Since $Z({}_R R)$ does not contain nonzero idempotents, $Z({}_R R)$ is small in R . Thus $Z({}_R R) \subseteq J(R)$. Similarly $Z(R_R) \subseteq J(R)$.

The next lemma is a key result for the proof of Theorem 1.

Lemma 6. Let R be a semiprimary left quasi-continuous ring. Then R is left continuous.

Proof. Since R is semiperfect, we can write $R = \bigoplus_{i=1}^n Re_i$ a direct sum of principal indecomposable left ideals Re_i . Each e_i is a nonzero primitive idempotent. By Lemma 1, each Re_i is quasi-continuous. Since an indecomposable CS-module is necessarily uniform, each Re_i is uniform. Since R is semiprimary, $\text{Soc}(Re_i)$ is essential in Re_i , and hence each Re_i has a unique minimal left ideal.

Without loss of generality we may assume that $\mathcal{E} = \{e_1, \dots, e_m\}$ is a basic set of primitive idempotents for R . Let $T_i = \text{Soc}(Re_i)$, $1 \leq i \leq m$. Hence $E(T_i) = E(Re_i)$.

Claim. $T_i J e_k = 0$, for every i and k , $1 \leq i, k \leq m$.

Suppose $k \neq i$ and $T_i a e_k \neq 0$ for some $a \in J$. It is easy to see that $T_i \cong T_i a e_k$ as minimal left ideals. Since $T_i a e_k \subseteq Re_k$, we infer that $T_k = T_i a e_k$. Since $T_i \cong T_k$, $E(Re_i) \cong E(Re_k)$. Now an application of Lemma 2 ensures that $Re_i \cong Re_k$, and hence $i = k$, a contradiction. Now, if $k = i$ and $T_i a e_i \neq 0$ for some $a \in J$, then $T_i a e_i \cong T_i$, and hence $T_i = T_i a e_i$ whence $T_i J e_i = T_i$.

Let $e = e_1 + \cdots + e_m$ be the basic idempotent. Then $T_i J e = T_i J e_i = T_i$. Thus $T_i J = T_i R$. Since R is semiprimary, $J^{n-1} \neq 0$ and $J^n = 0$ for some $n \geq 1$. Thus $(T_i J)(J^{n-1}) = (T_i R)(J^{n-1})$, and hence $T_i J^{n-1} = 0$. By repeating the argument, if necessary, we get $T_i J = 0$. Thus $T_i R = 0$, and hence $T_i = 0$, a contradiction. Thus $T_i J e_k = 0$ in every case, proving the claim.

Since $T_i J e_k = 0$ and $T_i J e = 0$, we infer that $T_i \subseteq l_R(J)$, $1 \leq i \leq m$. Since R/J is semisimple, $l_R(J) = \text{Soc}(R_R)$, and hence $\text{Soc}(R_R) \subseteq \text{Soc}(R_R)$. Now since $\text{Soc}_R R$ is essential in ${}_R R$ and $\text{Soc}(R_R) \subseteq l(J)$, it follows that $J(R) \subseteq Z({}_R R)$. By Lemma 5, we get $J(R) = Z({}_R R)$. And, by Lemma 3, R is left continuous.

Now the proof of Theorem 1 is an easy consequence of Lemma 6.

Proof of Theorem 1. By Lemma 6, R is a two-sided continuous ring, and, by Lemma 4, R is a quasi-Frobenius ring.

Corollary 1. *Let R be a left quasi-continuous ring with DCC on essential left ideals. Then R is a left continuous ring.*

Proof. By [1, Proposition 1.1], $R/\text{Soc}({}_R R)$ is left artinian. And, by [2, Corollary 6], R is left artinian. Then it follows from Lemma 6 that R is left continuous.

Corollary 2. *Let R be a two-sided quasi-continuous ring with DCC on essential left ideals. Then R is quasi-Frobenius.*

Remark. There is an example of a commutative, local, semiprimary, continuous ring which is not injective (see Rizvi [6]).

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