EXISTENCE OF π_1 -NEGLIGIBLE EMBEDDINGS IN 4-MANIFOLDS: A CORRECTION TO THEOREM 10.5 OF FREEDMAN AND QUINN

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ABSTRACT. The purpose of this note is to provide a correction to the existence part of Theorems 10.3 and 10.5 of *Topology of 4-manifolds*, Princeton Univ. Press, Princeton, NJ, 1990, which analyze when one can find a connected sum decomposition of a 4-manifold or a π_1 -negligible embedding in a 4-manifold respectively. In particular this gives a correction to the definition of the 4-dimensional Kervaire-Milnor invariant. We also define this invariant in a slightly more general context.

INTRODUCTION

Following [FQ] consider the following situation. Let $(V; \partial_0, V, \partial_1 V)$ be a compact 4-manifold triad with $\pi_1(V, \partial_0 V) = \{1\} = \pi_1(V, \partial_1 V)$ for all base points; each component of V has nonempty intersection with $\partial_1 V$, and components of V disjoint from $\partial_0 V$ are 1-connected. Suppose that $(W, \partial W)$ is a connected 4-manifold and that we are given a map $h: V \rightarrow W$ which restricts to an embedding of $\partial_0 V$ in ∂W . We wish to determine whether h is homotopic rel boundary to a π_1 -negligible embedding, that is, an embedding $f: V \rightarrow W$ such that the inclusion map $\pi_1(W - f(V)) \rightarrow \pi_1 W$ is an isomorphism. A few algebraic conditions must clearly be met if such an embedding is to exist. We must have algebraic duals in W to the generators of $h_* H_2(V, \partial_0 V; \mathbb{Z}[\pi_1 W])$, and h must preserve intersection and self-intersection numbers. (Both these hypotheses require more discussion which is supplied by [FQ, p. 169].)

To avoid repeating these hypotheses call such a situation a π_1 -negligible embedding problem. (This terminology is reasonable since by [FQ, Proposition 10.5B] most of these conditions are necessary.) As a special case if $V = (M_0; \phi, S^3)$ where M_0 is a 1-connected closed 4-manifold M with an open disk removed, this reduces to trying to find a decomposition of W as a connected sum with M , say $W \simeq M \# W'$, inducing a fixed decomposition of $\pi_2 W$. The solution to this question (and the related uniqueness question) is used in [FQ] to give a novel approach to the classification of closed 4-manifolds with $\pi_1 = \{1\}$ [FQ, Theorem 10.1] and $\pi_1 \mathbb{Z}$ [FQ, Theorem 10.7]. The correction we will discover concerns elements of order 2 in $\pi_1 W$ and hence does not

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affect these results. There is also a correction necessary to the uniqueness results of [FQ] which is given in [S]. This correction does slightly alter the classification for $\pi_1 = \mathbf{Z}$.

To define this correction precisely call a map $h: V \rightarrow W$ s -characteristic (s standing for spherically) if the second Stiefel-Whitney class $\omega_2: \pi_2 W \rightarrow \mathbf{Z}/2$ does not vanish but it does vanish on the subspace of $\pi_2 W$ perpendicular (in the sense of the intersection pairing) to $h_* H_2(V, \partial_0 V; \mathbf{Z}[\pi_1 W])$. If V is s -characteristic, let $\beta \in h_* H_2(V, \partial_0 V; \mathbf{Z}[\pi_1 W]) \subseteq H_2(W, \partial_0 V; \mathbf{Z}[\pi_1 W])$ be an s -characteristic class. Let $g \in \pi_1 W$ satisfy $g^2 = 1$. Let $f: \mathbf{RP}^2 \rightarrow W$ be an immersion representing g , i.e., with $f_* \pi_1 \mathbf{RP}^2 = \{1, g\}$. Define $\phi(g) \in \mathbf{Z}/2$ by $\phi(g) = \omega_2(f^* TW) + \beta \cdot f_* [\mathbf{RP}^2]$. To see that this is independent of the choice of β , we first note that any other s -characteristic class β' must have its mod 2 intersection numbers with spherical classes the same as those of β . There are dual 2-spheres in W to the generators of $h_* H_2(V, \partial_0 V; \mathbf{Z}[\pi_1 W])$. Therefore, β' must agree with β in $h_* H_2(V, \partial_0 V; \mathbf{Z}/2)$. Since only mod 2 intersection numbers with β enter into the formula, it is independent of β (although it may depend on h). Also note that it is independent of f since any other map of $\mathbf{RP}^2$ representing h differs from f by an element of $\pi_2 W$. From the definition it is clear that $\phi(1) = 0$.

With the terminology above the existence half of Theorem 10.5 can be phrased as follows.

Theorem 10.5. *Suppose $h: (V, \partial_0 V, \partial_1 V) \rightarrow W$ is a π_1 -negligible embedding problem with $\pi_1 W$ good. If V is not s -characteristic then h is homotopic rel boundary to an embedding. If V is s -characteristic then there is an obstruction $\text{km}(h) \in \mathbf{Z}/2$ which vanishes if and only if h is homotopic rel boundary to an embedding. If $*h: (V; \partial_0 V, \partial_1 V) \rightarrow *W$ is the canonical embedding problem then $\text{km}(*h) = \text{km}(h) + 1$.*

If [FQ] the last two statements are phrased as “ h is homotopic to such an embedding in exactly one of W or $*W$ ”. The phrasing here is in part intended to make the error more noticeable. The invariant $\text{km}(h)$ is combinatorically defined [FQ, pp. 178–183], and the proof that it is well defined is also combinatorial. Unfortunately one possible rearrangement is omitted in the proof, and in fact $\text{km}(h)$ is not quite well defined. Hence there are examples where V is s -characteristic but embeds in both W and $*W$.

Correction. *If V is s -characteristic and there is a $g \in \pi_1 W$ with $g^2 = 1$, $\omega_1(g) = 0$, and $\phi(g) = 1$, then $\text{km}(h)$ is not well defined and hence an embedding homotopic to h exists. If V is not s -characteristic or no such g exists then the statement above is correct.*

Interestingly this correction can be rephrased in a way more analogous to the 5-dimensional Kervaire-Milnor invariant [S], which arises from a correction to the uniqueness part of Theorem 10.5. If W is not connected then the km invariant should be viewed as an element of $H_0(W; \mathbf{Z}/2)$. Let Γ_0 be the subgroup of $H_0(W; \mathbf{Z}/2)$ generated by the components that contain a loop g with $g^2 = 1$, $\omega_1(g) = 0$, and $\phi(g) = 1$. Then the correction says that km actually takes values in $H_0(W; \mathbf{Z}/2)/\Gamma_0$. The 5-dimensional Kervaire-Milnor obstruction takes values in $H_1(W; \mathbf{Z}/2)/\Gamma_1$ where Γ_1 is the subgroup of $H_1(W; \mathbf{Z}/2)$ generated by loops g with $g^2 = 1$, $\omega_1(g) = 1$, and $\phi(g) = 1$.

Due to the logical structure of chapter 10 of [FQ] this correction leads to a correction to the existence half of Theorem 10.3.

Theorem 10.3. (1) Suppose M is a closed 1-connected 4-manifold and W is a 4-manifold with good fundamental group. Let $h: (\pi_2 M) \otimes \mathbb{Z}[\pi_1 W] \rightarrow \pi_2 W$ be a $\pi_1 W$ monomorphism which preserves the intersection and self-intersection forms λ and $\tilde{\mu}$. If the image is not s -characteristic then there is a decomposition $W \simeq M \# W'$ inducing the given decomposition of π_2 . If the image is s -characteristic then there is an obstruction $\text{km}(h) \in H_0(W; \mathbb{Z}/2)/\Gamma_0$ which vanishes if and only if such a decomposition exists.

A proof of these corrections would require repeating several pages of material from [FQ] almost verbatim. Instead we will prove the following closely related result which is the obvious generalization of the context for the original Kervaire-Milnor invariant [KM]. (Much of this material is still in [FQ]). The reader familiar with [FQ] will find it easy to use the appropriate section of the proof below to patch the proof in [FQ].

Theorem. Let W be a closed 4-manifold and $\beta \in \pi_2 W$ an s -characteristic class. Then there are two obstructions to finding a (locally flat) embedding $S^2 \rightarrow W$ homotopic to β :

- (1) the self-intersection number $\tilde{\mu}(\beta) \in \mathbb{Z}[\pi_1 W]/\mathbb{Z} \oplus \langle g - \bar{g} \rangle$,
- (2) a secondary obstruction $\text{km}(\beta) \in H_0(W; \mathbb{Z}/2)/\Gamma_0$.

It should be noted that Theorem 10.5 implies that if $\pi_1 W$ is good and an algebraic dual 2-sphere to β exists, then vanishing of these obstructions guarantees the existence of such an embedding.

Proof. Clearly the self-intersection number $\tilde{\mu}(\beta)$ is an obstruction to representing β by an embedding. Suppose therefore that it vanishes. Represent β by an immersion $\beta: S^2 \rightarrow W$. Since $\tilde{\mu}(\beta) = 0$, we may (after possibly introducing some extra kinks) pair up the self-intersections of β with immersed Whitney disks B_i , $1 \leq i \leq n$, transverse to β . Actually it is convenient to insist that they be paired only by “weak” Whitney disks where we allow the boundary arcs to be only immersed arcs in $\beta(S^2)$ and allow the framing to be incorrect. Let $\omega(B_i)$ be 0 if the framing of B_i is correct modulo 2 and 1 if it is not. (Hence ω is essentially a relative second Stiefel-Whitney class.) Define $\text{km}(B_i)$ to be the number intersections of $\text{int}(B_i)$ with $\beta(S^2)$ plus $\omega(B_i)$ modulo 2. Define $\text{km}(\beta)$ to be the sum of the $\text{km}(B_i)$ plus the number of self-intersections of the boundary arcs modulo 2.

We need only check that this is well defined. If it is, then it is clearly an obstruction to embedding. First suppose we alter β by a regular homotopy. Regular homotopies of 2-spheres in a 4-manifold are generated by Whitney moves, finger moves, and isotopies. The first two moves are inverse to each other, hence it suffices to check that the definition is invariant under finger moves. But a finger move introduces a new pair of intersections and an embedded Whitney disk. This Whitney disk contributes zero to $\text{km}(\beta)$. Hence we need only show $\text{km}(\cdot)$ does not depend on the choice of Whitney disks.

Suppose we alter the Whitney disks by a regular homotopy. If we are moving the interior of B_i only, then intersection points appear and disappear in pairs leaving the mod 2 invariant unchanged. If we move a neighborhood of

FIGURE 1. Pushing a boundary arc across a self-intersection of $\beta(S^2)$.

the boundary arcs without crossing the self-intersections of $\beta(S^2)$, then intersections of the boundary arcs appear and disappear in pairs. If we move a neighborhood of the boundary arcs across a self-intersection of $\beta(S^2)$, then we introduce (or remove) one new boundary intersection and introduce (or remove) one new intersection of the interior of the weak Whitney disk with $\beta(S^2)$ (see Figure 1). Finally if we do a twist around the boundary, then we introduce one new intersection of the interior of a weak Whitney disk with $\beta(S^2)$ and we change the framing by 1. In any of these situations the net change in km is zero mod 2.

Suppose now that we change the homotopy class of a weak Whitney disk. We may assume this is done by taking an immersed 2-sphere C in W and tubing it into the Whitney disk. The framing of the weak Whitney disk is changed by $C \cdot C$, and the number of intersections of the interior with β is changed by $C \cdot \beta(S^2)$. Since β is s -characteristic $C \cdot C + C \cdot \beta(S^2)$ is even and km is unchanged. Hence km is independent of the choice of Whitney disks. Also since any regular homotopy of the boundary arcs extends over the Whitney disk, it is independent of the boundary arcs. Furthermore, from now on we may assume that all Whitney disks are correctly framed with embedded boundary since we could always reduce to this case by the results above.

Hence km depends only on the pairing of the intersection points. There are two operations that can change this pairing. First suppose we have two pairs of intersection points. Say they are the images of w and w' , x and x' , y and y' , and z and z' , and suppose that w , x , y , and z all represent the same element of $\pi_1 W$ and have alternating signs. (There is a minor error in [FQ] here; since we have not assumed W orientable, we cannot talk about the signs of the intersection points. However the preimages get well-defined signs regardless.) Suppose we have Whitney disks B_1 and B_2 whose boundaries are arcs joining w to x , w' to x' , y to z , and y' to z' . Suppose C_1 is a Whitney disk whose boundary arcs join x to y and x' to y' . Then a (correctly framed) Whitney disk C_2 whose boundary arcs join w to z and w' to z' can be constructed from parallel copies of B_1 , B_2 , and C_1 by adding small twists near $\beta(x)$ and $\beta(y)$ (see Figure 2). Then $\text{km}(C_2) \equiv \text{km}(B_1) + \text{km}(B_2) + \text{km}(C_1)$ since the intersections of C_2 with β are the sum of those of the other three Whitney disks.

By the move above we may obtain any pairing of the intersection points. However, there is still one more possibility. Suppose we have two intersection points. Say they are the images of x and x' and y and y' , and suppose that x and y represent an element $g \in \pi_1 W$ with $g^2 = 1$. Suppose B is a Whitney disk whose boundary is arcs from x to y and x' to y' . Then we

FIGURE 2. Whitney disks for the four intersection points.

FIGURE 3. Attaching curves for the two Whitney disks.

could attempt to choose a new Whitney disk whose boundary is arcs from x to y' and x' to y . If $\omega_1(g) = 0$, then the signs of the intersection points are correct and there is such a C . This is the operation that was omitted in [FQ].

We need only decide when this changes $\text{km}(\beta)$. Choose boundary arcs as shown in Figure 3. Then the union of the region Σ bounded by them and the two Whitney disks B and C is an immersed $\mathbf{RP}^2$ representing g . We can therefore use this $\mathbf{RP}^2$ to calculate $\phi(g)$. There are four contributions to $\phi(g)$: the second Stiefel-Whitney class of TW restricted to this $\mathbf{RP}^2$, the number of intersections of B and C with $\beta(S^2)$, and the number of intersections of Σ with $\beta(S^2)$ created when we push our $\mathbf{RP}^2$ off $\beta(S^2)$. The intersections of B and C with $\beta(S^2)$ contribute $\text{km}(B)$ and $\text{km}(C)$ respectively. Since B and C are correctly framed, TW pulls back to a trivial 4-dimensional bundle over this $\mathbf{RP}^2$; hence, an even number of intersections with $\beta(S^2)$ are introduced by the push off. One noncomputational way to see this is to note these two terms can be calculated from the standard model with B and C embedded since intersections do not affect them. Then note that we can produce such a model in $\mathbf{R}^4$ where both contributions are clearly trivial. Therefore, $\phi(g) = \text{km}(B) + \text{km}(C)$. Hence $\text{km}(\beta)$ is unchanged if $\phi(g) = 0$ and changed if $\phi(g) = 1$.

As the following lemma illustrates in some cases the Kervaire-Milnor invariant has a noncombinatorial definition. A more general version would be interesting.

Lemma. *Let W be a closed 4-manifold with $H_2(\pi_1 W; \mathbf{Z}/2) = 0$ and β an s -characteristic class with $\mu(\beta) = 0$. Then*

$$\text{km}(\beta) \equiv (\beta \cdot \beta - \sigma(W))/8 + KS(W) \pmod{2}.$$

Proof. Since $H_2(\pi_1 W; \mathbf{Z}/2) = 0$ every class in $H_2(W; \mathbf{Z}/2)$ is spherical and β is characteristic. Fix an immersed 2-sphere representing β and immersed

Whitney disks for the self-intersections. Denote this 2-complex by K , and fix a spin structure on $W - \beta(S^2)$. Do surgery (with the framing given by the spin structure) on a collection of loops in $W - K$ to kill off $\pi_1 W$. This does not change any of the terms above since K is unaffected. Thus we may assume W is 1-connected. Replacing W by $W \# \|E_8\|$ if necessary we may also assume $KS(W) = 0$. After further connected sums with copies of $S^2 \times S^2$ we may assume W is smooth. The result now follows from the geometric proof of Rochlin's theorem in [FK].

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