GOLDIE'S THEOREM FOR ALTERNATIVE RINGS

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ABSTRACT. It is known that the socle of a semiprime Goldie ring is generated by a central idempotent and that a prime Goldie ring with a nonzero socle is a simple artinian ring. We prove the extension of these results to alternative rings. We also give an analogue of Goldie's theorem for alternative rings. A Goldie-like theorem was obtained earlier by the authors for noetherian alternative rings by a quite different method.

1. INTRODUCTION

By utilizing some results of Slater and the classical Goldie theorem, a Goldie-like theorem was proved in [3] for semiprime noetherian alternative rings. In this paper, by considering the extension to alternative rings of the classical methods of the associative theory, we find again this result for a large class of alternative rings, which includes all semiprime (associative) Goldie rings. In this new approach we do not need Goldie's theorem; however a single result of Slater is needed. Namely, if I is a right ideal of a semiprime alternative ring R , then $3I \subseteq N(R)$ or $I \cap Z(R) \neq 0$, where $N(R)$ is the nucleus of R and $Z(R)$ is its center.

It is known (see [2]) that the socle of a semiprime Goldie ring is generated by a central idempotent and that a prime Goldie ring with nonzero socle is a simple artinian ring. We also give the extension to alternative rings of these results.

2. DEFINITIONS AND NOTATION

In any nonassociative (i.e., not necessarily associative) ring R the associator of three elements $a, b, c \in R$ is defined by $(a, b, c) = (ab)c - a(bc)$ and the commutator of two elements $a, b \in R$ by $[a, b] = ab - ba$. The nucleus $N = N(R)$ is defined by

$$N = \{a \in R \mid (a, R, R) = (R, a, R) = (R, R, a) = 0\}$$

and the center $Z = Z(R)$ by

$$Z = \{z \in N \mid [z, R] = 0\}.$$

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R is alternative provided the following identity holds:

$$(a, a, b) = 0 = (b, a, a).$$

An alternative ring R is said to be prime if for all ideals I, J , of R , $IJ = 0$ implies $I = 0$ or $J = 0$; it is said to be semiprime if for every ideal I of R , $I^2 = 0$ implies $I = 0$.

The (right) socle of an alternative ring R is the sum of all the minimal right ideals of R (with $\text{soc}(R) = 0$ if R has no minimal right ideals).

A right ideal A of an alternative ring R is said to be essential if for every right ideal B of R , $A \cap B = 0$ implies $B = 0$.

If X is a nonempty subset of an alternative ring R , then the right annihilator of X in R is

$$r(X) = \{a \in R \mid xa = 0 \text{ for all } x \in X\}.$$

The left annihilator of X is

$$l(X) = \{a \in R \mid ax = 0 \text{ for all } x \in X\}.$$

It is easily seen that if X is a left (right) ideal of R then $r(X)$ ($l(X)$) is a right (left) ideal of R . If X is an ideal of R then $r(X), l(X)$ are ideals of R and for every subset X we have $r(l(r(X))) = r(X)$ and $l(r(l(X))) = l(X)$. If $a \in N(R)$ then it is obvious that $r(a)$ is a right ideal and $l(a)$ is a left ideal.

3. THE SOCLE OF A SEMIPRIME ALTERNATIVE RING IN THE PRESENCE OF SOME CHAIN CONDITIONS

It has been proved in [4] that if A is a right ideal of a semiprime alternative ring R then $3A \subseteq N(R)$ or $A \cap Z(R) \neq 0$. This implies the following lemma.

Lemma 1. *If in a semiprime alternative ring R , $3(a, b, c) = 0$ implies $(a, b, c) = 0$, for every associator (a, b, c) in R , then, for every right ideal A of R , $A \subseteq N(R)$ or $A \cap Z(R) \neq 0$.*

Lemma 2. *Suppose that a semiprime alternative ring R satisfies the following conditions:*

- (i) $3(a, b, c) = 0$ implies $(a, b, c) = 0$, for every associator (a, b, c) in R .
- (ii) R has the a.c.c. for the right annihilators of the form $r(a)$, $a \in N(R)$.

Then any nonzero one-sided ideal of R contains a nonnilpotent element of $N(R)$.

Proof. Let I be a nonzero one-sided ideal of R and suppose that any element of $I \cap N(R)$ is nilpotent. Since R is semiprime, $I \cap Z(R) = 0$. By Lemma 1, $I \subseteq N(R)$. If $0 \neq a_1 \in I$, then there exists an element x of R such that $a_2 = a_1 x a_1 \neq 0$. It is clear that $r(a_1) \subseteq r(a_2)$. We will show that we have a strict inclusion. Suppose first that I is a right ideal. Then $0 \neq a_1 x \in I$. There exists $n \geq 1$ such that $(a_1 x)^n \neq 0$ and $(a_1 x)^{n+1} = 0$. If $n = 1$ then $x \in r(a_2)$ and $x \notin r(a_1)$. Suppose that $n \geq 2$. In this case $x(a_1 x)^{n-1} \in r(a_2)$ and $x(a_1 x)^{n-1} \notin r(a_1)$.

Suppose now that I is a left ideal. There exists $n \geq 1$ such that $(x a_1)^n \neq 0$ and $(x a_1)^{n+1} = 0$. If $n = 1$ then $x a_1 \in r(a_2)$ and $x a_1 \notin r(a_1)$. Suppose that $n \geq 2$. If $a_1(x a_1)^n = 0$ then $(x a_1)^{n-1} \in r(a_2)$ and $(x a_1)^{n-1} \notin r(a_1)$. If not then $(x a_1)^n \in r(a_2)$ and $(x a_1)^n \notin r(a_1)$. We have proved that $r(a_1) \subsetneq r(a_2)$.

Repeating this argument we can find $0 \neq a_3 \in I$ such that $r(a_1) \subsetneq r(a_2) \subsetneq r(a_3), \dots$. Contradiction.

Corollary 1. *Let R be as in Lemma 2. Then R has no nonzero nil one-sided ideals.*

Corollary 2 (Zhevlakov). *Let A be an alternative noetherian algebra over a ring F containing an element $1/3$. Then any one-sided nil ideal of A is nilpotent.*

Proof. Let I be a one-sided nil ideal in A . By Corollary 1, $I \subseteq B(A)$, where $B(A)$ is the Baer radical of A . Since $B(A)$ is nilpotent [7], I is nilpotent.

Corollary 3. *Let R be as in Lemma 2 and let K be an essential right ideal of R . Then $l(K) = 0$.*

Proof. Let a be in $l(K) \cap N(R)$. Since $r(a) \subseteq r(a^2) \subseteq \dots$, there exists $n \geq 1$ such that $r(a^n) = r(a^{n+1}) = \dots$. We have $K \subseteq r(a)$, so that $r(a^n)$ is an essential right ideal of R . Now $a^n R \cap r(a^n) = 0$ which implies that $a^n R = 0$. Since R is semiprime, $a^n = 0$. Thus $l(K) = 0$ by Lemma 2.

Lemma 3. *Suppose that an alternative ring R contains no infinite direct sums of nonzero right ideals and that c is a right regular element of R with $c \in N(R)$. Then cR is an essential right ideal of R .*

Proof. Let I be a right ideal of R such that $I \cap cR = 0$. Since $c \in N(R)$, $c^n I$ is a right ideal of R for every n . It is easy to verify that the sum $I + cI + c^2 I + \dots$ is direct. Thus there exists $n \geq 1$ such that $c^n I = 0$. Therefore $I = 0$.

From Lemma 3 and Corollary 3 we obtain the following lemma.

Lemma 4. *Suppose that a semiprime alternative ring R satisfies the following conditions:*

- (i) $3(a, b, c) = 0$ implies $(a, b, c) = 0$, for every associator (a, b, c) in R .
- (ii) R has the a.c.c for the right annihilators of the form $r(a)$, $a \in N(R)$.
- (iii) R contains no infinite direct sums of nonzero right ideals.

Then every element of $N(R)$ which is right regular in R is regular.

Theorem 1. *Let R be as in Lemma 4 and let K be an essential right ideal of R . Then K contains an element of $N(R)$ which is regular in R .*

Proof. Let a_1 be a nonnilpotent element of $K \cap N(R)$ with $r(a_1)$ as large as possible. We have $r(a_1) \subseteq r(a_1^2)$ and a_1^2 is a nonnilpotent element of $K \cap N(R)$. Hence $r(a_1) = r(a_1^2)$. If $r(a_1) = 0$ we stop. If not, $r(a_1) \cap K \neq 0$. Let a_2 be a nonnilpotent element of $r(a_1) \cap K \cap N(R)$ with $r(a_2)$ as large as possible. Then $r(a_2) = r(a_2^2)$. It is easy to verify that the sum $a_1 R + a_2 R$ is direct and $r(a_1 + a_2) = r(a_1) \cap r(a_2)$. If $r(a_1 + a_2) = 0$ we stop. If not let a_3 be a nonnilpotent element of $r(a_1 + a_2) \cap K \cap N(R)$ with $r(a_3)$ as large as possible. Then $r(a_3) = r(a_3^2)$ and the sum $a_1 R + a_2 R + a_3 R$ is direct. Thus $r(a_1 + a_2 + a_3) = r(a_1) \cap r(a_2) \cap r(a_3)$. Because R contains no infinite direct sums of nonzero right ideals, this process must stop, and when it does there are elements $a_1, \dots, a_n$ of $K \cap N(R)$ such that $r(a_1 + \dots + a_n) = 0$. Therefore $a = a_1 + \dots + a_n$ is an element of $K \cap N(R)$ which is right regular in R . By Lemma 4, a is regular in R .

Theorem 2. *Let R be as in Lemma 4 and let S be the socle of R . Then there is a central idempotent element e of R such that $S = eR$.*

Proof. There is a right ideal T of R such that $S \cap T = 0$ and $S \oplus T$ is an essential right ideal of R . By Theorem 1, there exist $s \in S$ and $t \in T$ such that $c = s + t \in N(R)$ and c is a regular element of R . Since S is an ideal of R [6] and $S \cap T = 0$, $TS = 0$. Therefore $cS \subseteq S$. Suppose that $cS \neq S$. Then for every $n \geq 0$ there exist a nonzero right ideal A_n of R such that $c^n S = A_n \oplus c^{n+1} S$. It is easy to see that the sum $A_0 + A_1 + A_2 + \dots$ is direct, which contradicts condition (iii). Thus $cS = S$. Therefore there is $e \in S$ such that $s = ce$. We have $ce = se + te = se = ce^2$ so that $e = e^2$. Also, for any $x \in S$ we have $cx = sx + tx = sx = (ce)x = c(ex)$ so that $x = ex$. Thus $S = eR$. We will show that $e \in N(R)$. Let $x, y \in R$, we have $(c, x, y) = 0 = (s, x, y) + (t, x, y)$. Since $T \cap S = 0$, $(s, x, y) = 0$. Therefore $s \in N(R)$. We have $c(e, x, y) = (ce, x, y) = (s, x, y) = 0$. Thus $e \in N(R)$. To show that $e \in Z(R)$, let $x \in R$. Then $xe \in S$ and $S = eR$ so that $xe = exe$. Now the equality $(exe - ex)eR = 0$ implies that $exe - ex \in S \cap l(S) = 0$ because R is semiprime. Hence $ex = exe$.

Theorem 3. *Suppose that a prime alternative ring R satisfies the conditions (i)–(iii) of Lemma 4 and has a nonzero socle. Then R is simple and artinian.*

Proof. Let I be a nonzero ideal of R and let A be a right ideal of R such that $I \cap A = 0$. Since $AI \subseteq A \cap I$, $AI = 0$. It is easy to see that $A + RA$ is the ideal generated by A in R and $(A + RA)I = 0$. Thus $A = 0$ because R is prime. Therefore every nonzero ideal of R is essential as a right ideal. This implies that $S = \text{soc}(R) \subseteq I$. By Theorem 1, S contains a regular element c with $c \in N(R)$. As in the proof of Theorem 2 we have $S = cS$. Since $cR \subseteq S$, $cR = cS$. Hence $R = S$. Thus $I = R$ so that R is simple. Also, the equality $R = S$ and condition (iii) shows that R is a right artinian alternative ring so that R is artinian.

4. MAIN THEOREM

Throughout this section R will denote an alternative ring and S the subset of R defined by

$$S = \{a \in N(R) \mid a \text{ is regular in } R\}.$$

Suppose that S is nonempty. An alternative ring Q with 1 is called a right quotient ring of R relative to S if

1. R is a subring of Q ,
2. every element of S is invertible in Q ,
3. each element $q \in Q$ can be written in the form $q = xa^{-1}$, where $x \in R$ and $a \in S$.

Lemma 5. *If Q is a right quotient ring of R relative to S then $N(R) \subseteq N(Q)$ and for every $a \in S$, $a^{-1} \in N(Q)$.*

Proof. Any nonassociative ring satisfies the identity

$$(1) \quad x(y, z, t) + (x, y, z)t = (xy, z, t) - (x, yz, t) + (x, y, zt)$$

and any alternative ring satisfies the identity [1]

$$(2) \quad (xy, z, t) - y(x, z, t) - (y, z, t)x = ([x, y], z, t) + (x, y, [z, t]).$$

Let $a \in S$ and $x, y \in R$. By (2),

$$\begin{aligned} (a^{-1}, x, y)a &= (aa^{-1}, x, y) - a^{-1}(a, x, y) \\ &\quad - ([a, a^{-1}], x, y) - (a, a^{-1}, [x, y]) \\ &= 0. \end{aligned}$$

Thus

$$(3) \quad (a^{-1}, x, y) = 0, \quad a \in S, \quad x, y \in R.$$

Let $n \in N(R)$, $a \in S$, $x, y \in R$. By (1) and (3),

$$(n, x, ya^{-1}) = n(x, y, a^{-1}) + (n, x, y)a^{-1} - (nx, y, a^{-1}) + (n, xy, a^{-1}) = 0.$$

We have proved that

$$(4) \quad (n, x, q) = 0, \quad n \in N(R), \quad x \in R, \quad q \in Q.$$

Let $a \in S$, $q \in Q$, $x \in R$. By (1) and (4)

$$a(a^{-1}, q, x) = (aa^{-1}, q, x) - (a, a^{-1}q, x) + (a, a^{-1}, qx) - (a, a^{-1}, q)x = 0.$$

Thus

$$(5) \quad (a^{-1}, q, x) = 0, \quad a \in S, \quad q \in Q, \quad x \in R.$$

Let $n \in N(R)$, $a \in S$, $x \in R$, $q \in Q$. We have

$$(n, q, xa^{-1}) = n(q, x, a^{-1}) + (n, q, x)a^{-1} - (nq, x, a^{-1}) + (n, qx, a^{-1}) = 0$$

by virtue of (4) and (5). Thus

$$(6) \quad (n, q, q') = 0, \quad n \in N(R), \quad q, q' \in Q.$$

Then $N(R) \subseteq N(Q)$. Let $a \in S$ and $q, q' \in Q$. Then $a(a^{-1}, q, q') = (aa^{-1}, q, q') = 0$. It follows that $a^{-1} \in N(Q)$.

Lemma 6. *The right quotient ring of R relative to S , if it exists, is unique in the sense that any two such rings are isomorphic under a homomorphism which restricts to the identity on R .*

Proof. Suppose that Q_1 and Q_2 are two right quotient rings relative to S of R . Let $x, y \in R$ and $a, b \in S$. It follows from Lemma 5 that if the equality $xa^{-1} = yb^{-1}$ is valid in Q_1 then it is valid in Q_2 . Consider $\sigma: Q_1 \rightarrow Q_2$ such that $\sigma(xa^{-1}) = xa^{-1}$, where $x \in R$ and $a \in S$. It is easy to see that σ is the desired homomorphism.

The next lemma can be proved by the same method used for associative rings.

Lemma 7. *R has right quotient ring relative to S if and only if it satisfies the right Ore condition with respect to S ; i.e., given $x \in R$, $a \in S$ there exists $y \in R$, $b \in S$ such that $xb = ay$.*

Main Theorem. *Suppose that R is semiprime and satisfies the following conditions.*

- (i) $3(a, b, c) = 0$ implies $(a, b, c) = 0$, for every associator (a, b, c) in R .
- (ii) R satisfies the a.c.c. for the right annihilators of the form $r(a)$, $a \in N(R)$.
- (iii) R contains no infinite direct sums of nonzero right ideals.

Then the right quotient ring of R relative to S exists and it is semisimple and artinian.

Proof. Since R is essential as a right ideal of R then by Theorem 1, $S \neq \emptyset$. We will prove that R satisfies the right Ore condition with respect to S . Let $x \in R$ and $c \in S$. Let I be a right ideal of R such that $xI \subseteq cR$. If I is essential then by Theorem 1, there exists $d \in I \cap S$ such that $xd = cy$, $y \in R$. Suppose that I is not essential. Therefore there exists a nonzero right ideal J of R such that $I \cap J = 0$. By Lemma 3, cR is an essential right ideal of R . Then $J \cap cR \neq 0$. In view of Lemma 1, $J \cap cR \cap Z(R) \neq 0$ or $J \cap cR \subseteq N(R)$. Suppose first that $J \cap cR \cap Z(R) \neq 0$. Let $0 \neq z \in J \cap cR \cap Z(R)$. Then $I_1 = zR$ is a nonzero ideal of R because R is semiprime. We have $I_1 \subseteq J \cap cR$ and $xI_1 = x(zR) = z(xR) \subseteq cR$. Also $I \cap I_1 \subseteq I \cap J = 0$. Thus $x(I \oplus I_1) \subseteq cR$.

Suppose now that $J \cap cR \subseteq N(R)$. In this case $x(J \cap cR)$ is a right ideal of R . If $x(J \cap cR) = 0$ then $x(I \oplus I_1) \subseteq cR$, where $I_1 = J \cap cR$. If not then $x(J \cap cR) \cap cR \neq 0$. Let $y \in J \cap cR$ with $0 \neq xy \in cR$. Then $I_1 = yR$ is a nonzero right ideal of R . We have $xI_1 = x(yR) = (xy)R \subseteq cR$ and $I \cap I_1 \subseteq I \cap J = 0$. Therefore $x(I \oplus I_1) \subseteq cR$.

In either case, there exists a nonzero right ideal I_1 of R such that $x(I \oplus I_1) \subseteq cR$. If $I \oplus I_1$ is essential we stop. If not, there exists a nonzero right ideal I_2 of R such that $x(I \oplus I_1 \oplus I_2) \subseteq cR$, and so on. Since R contains no infinite direct sums of nonzero right ideals, this process must stop, and when it does there is an essential right ideal K of R such that $xK \subseteq cR$. Thus R satisfies the right Ore condition with respect to S . By Lemma 7, the right quotient ring Q of R relative to S exists.

Let I be a right ideal of Q . It is easy to see that $I \cap R = 0$ implies $I = 0$. It follows that Q is semiprime. If I is a right ideal of Q , then there is a right ideal A of R such that $(I \cap R) \oplus A$ is an essential right ideal of R . Let $c \in S \cap ((I \cap R) \oplus A)$ so that $c = x + a$, where $x \in I \cap R$ and $a \in A$. Consider $J = AQ = \{bq | b \in A, q \in Q\} = \{bs^{-1} | b \in A, s \in S\}$, it is easy to see that J is the right ideal of Q generated by A . We have $1 = cc^{-1} = xc^{-1} + ac^{-1} = e + f$, with $e = xc^{-1} \in I$ and $f = ac^{-1} \in J$. Since $I \cap J = 0$, $Q = I \oplus J$. We have $fe = e - e^2 \in I \cap J = 0$. Then $e^2 = e$. Also $f^2 = f$. Let $q, q' \in Q$ then $(1, q, q') = 0 = (e + f, q, q') = (e, q, q') + (f, q, q')$. Thus $(e, q, q') \in I \cap J = 0$. Therefore $e \in N(Q)$. Also $f \in N(Q)$. Let $x \in I$, $x = ex + fx$ so that $fx = x - ex \in I \cap J = 0$. Then $x = ex$ for every $x \in I$. Therefore $I = eQ$. We have proved that every right ideal of Q is generated by an idempotent element e with $e \in N(Q)$.

Since for every right ideal I of Q there exists a right ideal J of Q such that $I \oplus J = Q$, if $I_1 \supseteq I_2$ are two right ideals of Q then there is a right ideal A of Q such that $I_1 = I_2 \oplus A$. Let $I_1 \supseteq I_2 \supseteq \dots$ be a strictly decreasing chain of right ideals of Q . For each $n \geq 1$ there is a nonzero right ideal A_n of Q such that $I_n = I_{n+1} \oplus A_n$. The sum $A_1 + A_2 + \dots$ is direct. Therefore $(A_1 \cap R) \oplus (A_2 \cap R) \oplus \dots$ is a direct sum of nonzero right ideals of R , a contradiction. Then Q is right artinian.

Let I be an ideal of Q then there exists $e \in N(Q)$ such that $e^2 = e$ and $I = eQ$. It is easy to see that $e \in Z(Q)$ and so there is an ideal J of Q such that $I \oplus J = Q$. It follows that Q is semisimple. Consequently, Q is semisimple and artinian.

Remark 1. It is obvious that condition (i) is satisfied in any associative ring.

Remark 2. Let R be as in the main Theorem and let Q be its right quotient ring relative to S . Let x be a regular element of R . It is easy to see that x is right regular in Q . By the structure theorem of semisimple artinian alternative rings [5, 7], it is easy to see that every right regular element of Q is invertible. Thus x is invertible in Q . Therefore Q is in fact the total right quotient ring of R .

Remark 3. As in the associative case, we can prove that if R has a right quotient ring Q relative to S which is semisimple and artinian then R is semiprime and satisfies conditions (ii) and (iii) of the main theorem.

Theorem 4. *Suppose that R is prime and satisfies conditions (i)–(iii) of the main theorem. Then R has a right quotient ring relative to S which is simple and artinian.*

Proof. Let Q be the right quotient ring of R relative to S and let I be a nonzero ideal of Q . Thus $I \cap R$ is a nonzero ideal of R . Since every nonzero ideal of R is essential as a right ideal (see the proof of Theorem 3), by Theorem 1, there exists an element $c \in S \cap (I \cap R)$. We have $1 = cc^{-1} \in I$. Therefore $I = Q$.

Corollary 4 (Zhevlakov). *Let A be an alternative noetherian algebra over a ring F containing $1/3$. Then any nil subring of A is solvable.*

Proof. Let B be a nil subring of A . Consider the algebra $\bar{A} = A/B(A)$, where $B(A)$ is the Baer radical of A . By the main theorem, $\bar{A}$ can be imbedded in an artinian algebra. Let $\bar{B}$ denote the image of B in $\bar{A}$ under the natural homomorphism. By a theorem of Zhevlakov [7], $\bar{B}$ is nilpotent. Since $B(A)$ is nilpotent [7], B is solvable.

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