INTEGRATION OF VECTOR-VALUED PSEUDO-ALMOST PERIODIC FUNCTIONS

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ABSTRACT. A necessary and sufficient condition is given to show that the indefinite integral of a vector-valued pseudo-almost periodic function is again pseudo-almost periodic. Then we use this result to answer a question about weakly almost periodic functions.

Throughout this paper, X denotes a Banach space and $\mathbb{J}_a$ stands for $[a, \infty)$ when $a \in \mathbb{R}$ and for $\mathbb{R}$ when $a = -\infty$; $\mathcal{C}(\mathbb{J}_a, X)$ denotes the space of all bounded continuous functions from $\mathbb{J}_a$ to X . Also, m denotes Lebesgue measure on $\mathbb{R}$.

Let $f \in \mathcal{C}(\mathbb{R}, X)$. The translate of f by $s \in \mathbb{R}$ is the function $R_s f(t) = f(t + s)$, $t \in \mathbb{R}$. f is called (weakly) almost periodic if the set $\{R_s f\} = \{R_s f : s \in \mathbb{R}\}$ is (weakly) relatively compact in $\mathcal{C}(\mathbb{R}, X)$ [3]. Denote by $(\mathcal{WAP}(\mathbb{R}, X))$ $\mathcal{AP}(\mathbb{R}, X)$ all such functions.

In case $X = \mathbb{C}$, we will omit X from our notation and write, for example, $\mathcal{C}(\mathbb{J}_a)$ for $\mathcal{C}(\mathbb{J}_a, X)$.

For a function $f \in \mathcal{AP}(\mathbb{R})$, the classical Bohl-Bohr integration theorem (cf. [2]) asserts that the indefinite integral $F(t) = \int_0^t f(u) du$ will also be almost periodic on $\mathbb{R}$ whenever F is bounded. For $f \in \mathcal{AP}(\mathbb{R}, X)$, there is Kadets's generalized Bohl-Bohr theorem (see [4]). A question arises naturally in $\mathcal{WAP}(\mathbb{R}, X)$; that is, if $f \in \mathcal{WAP}(\mathbb{R}, X)$, what is a necessary and sufficient condition for F to be again in $\mathcal{WAP}(\mathbb{R}, X)$?

Let $f \in \mathcal{WAP}(\mathbb{R}, X)$. Then $f = g + \varphi$, where $g \in \mathcal{AP}(\mathbb{R}, X)$ and $\varphi \in \mathcal{WAP}_0(\mathbb{R}, X)$, the subspace of $\mathcal{WAP}(\mathbb{R}, X)$ whose members have the zero function in the weak closure of the set of translates [3, Theorem 4.11]. The difficult part of answering the question is to handle the function φ . For this purpose we introduce a new generalization of almost periodic functions, which we call pseudo-almost periodic functions.

In this paper, we first set up some theorems of the indefinite integral of a pseudo-almost periodic function from $\mathbb{J}_a$ to X . Then we use these theorems to answer the question.

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Definition 1. A subset P of $\mathbb{J}_a$ is said to be relatively dense in $\mathbb{J}_a$ if there exists a number $l > 0$ such that $[t, t + l] \cap P \neq \emptyset$ ($t \in \mathbb{J}_a$).

A function $g \in \mathcal{C}(\mathbb{R}, X)$ is in $\mathcal{AP}(\mathbb{R}, X)$ if and only if, for $\epsilon > 0$, the set

$$P(\epsilon) = \{\tau \in \mathbb{R} : \|g(t + \tau) - g(t)\| < \epsilon \text{ for all } t \in \mathbb{R}\}$$

is relatively dense in $\mathbb{R}$ [2, Theorem 6.6]. A function $g \in \mathcal{AP}(\mathbb{R}, X)$ is uniformly continuous.

Set

$$\mathcal{PAP}_0(\mathbb{J}_a, X) = \left\{ \varphi \in \mathcal{C}(\mathbb{J}_a, X) : \lim_{t \rightarrow \infty} \frac{1}{t - a} \int_a^t \|\varphi(s)\| ds = 0 \right. \\ \left. \left(\lim_{t \rightarrow \infty} \frac{1}{2t} \int_{-t}^t \|\varphi(s)\| ds = 0, \text{ when } a = -\infty \right) \right\}.$$

Definition 2. A function $f \in \mathcal{C}(\mathbb{J}_a, X)$ is called pseudo-almost periodic if

$$f = g|_{\mathbb{J}_a} + \varphi,$$

where $g \in \mathcal{AP}(\mathbb{R}, X)$ and $\varphi \in \mathcal{PAP}_0(\mathbb{J}_a, X)$. Denote by $\mathcal{PAP}(\mathbb{J}_a, X)$ all such functions. As in [7], g and φ are called the almost periodic component and the ergodic perturbation of f , respectively.

Definition 3. A closed subset C of $\mathbb{J}_a$ is said to be an ergodic zero set in $\mathbb{J}_a$ if $m(C \cap [a, t])/(t - a) \rightarrow 0$ as $t \rightarrow \infty$ ($m(C \cap [-t, t])/2t \rightarrow 0$ as $t \rightarrow \infty$, when $a = -\infty$).

The proof of the following proposition is straightforward.

Proposition 4. A function $\varphi \in \mathcal{C}(\mathbb{J}_a, X)$ is in $\mathcal{PAP}_0(\mathbb{J}_a, X)$ if and only if, for $\epsilon > 0$, the set $C_\epsilon = \{t \in \mathbb{J}_a : \|\varphi(t)\| \geq \epsilon\}$ is an ergodic zero subset in $\mathbb{J}_a$.

Proposition 5. Let C be an ergodic zero set in $\mathbb{J}_a$. Then for any $\delta > 0$ and $L > 0$, there exists an interval $(u, v) \subset \mathbb{J}_a$ with the properties that $v - u > L$ and $m(C \cap (u, v)) < \delta$.

Proof. If such a (u, v) does not exist, one sees readily that

$$\liminf_{t \rightarrow \infty} m(C \cap [a, t])/t \geq \delta/2L$$

$$(\liminf_{t \rightarrow \infty} m(C \cap [-t, t])/2t \geq \delta/2L, \text{ when } a = -\infty).$$

Using Propositions 4 and 5, we can show that, for $g \in \mathcal{AP}(\mathbb{R}, X)$, if $g|_{\mathbb{J}_a} \in \mathcal{PAP}_0(\mathbb{J}_a, X)$, then $g = 0$. In fact, since g is uniformly continuous, for $\epsilon > 0$ there is $\delta > 0$ such that $\|g(t') - g(t'')\| < \epsilon$ whenever $t', t'' \in \mathbb{R}$ with $|t' - t''| < \delta$. Let $l > 0$ be the number for $P(\epsilon)$ as in Definition 1, let C_ϵ be the ergodic zero subset for $g|_{\mathbb{J}_a} \in \mathcal{PAP}_0(\mathbb{J}_a, X)$ in $\mathbb{J}_a$ as in Proposition 4, and let $L = 2l$. By Proposition 5, for $\delta > 0$ and $L > 0$, there exists an interval $(u, v) \subset \mathbb{J}_a$ with the properties that $v - u > L$ and $m(C \cap (u, v)) < \delta$. It follows that $\|g(t)\| < 2\epsilon$, $t \in (u, v)$. For $t \in \mathbb{R}$, since $v - u > 2l$, one can find $\tau \in P(\epsilon)$ such that $t + \tau \in (u, v)$. Therefore, $\|g(t)\| \leq \|g(t) - g(t + \tau)\| + \|g(t + \tau)\| < 3\epsilon$. Since $\epsilon > 0$ is arbitrary, $g = 0$. Thus

$$\mathcal{PAP}(\mathbb{J}_a, X) = \mathcal{AP}(\mathbb{R}, X)|_{\mathbb{J}_a} \oplus \mathcal{PAP}_0(\mathbb{J}_a, X).$$

To show the next theorem, we need the following lemma.

Lemma 6. *Let P be relatively dense in $\mathbb{J}_a$, and let C be an ergodic zero set in $\mathbb{J}_a$. Then, for any given interval $[c, d] \subset \mathbb{R}$ and $\delta > 0$, there exist $(u, v) \subset \mathbb{J}_a$ and $\tau \in P$ such that*

$$[c, d] + \tau \subset (u, v) \quad \text{and} \quad m(C \cap (u, v)) < \delta.$$

Proof. Let $l > 0$ be the number for P as in Definition 1, and let $L = l + (d - c)$. By Proposition 5, there exists an interval $(u, v) \subset \mathbb{J}_a$ such that $m(C \cap (u, v)) < \delta$ and $L < v - u$. Since we can assume that $u - c \in \mathbb{J}_a$, we can choose $\tau \in [u - c, u - c + l] \cap P$. If $t \in [c, d]$,

$$u < c + \tau \leq t + \tau \leq d + \tau \leq d + u - c + l < v;$$

that is, $[c, d] + \tau \subset (u, v)$.

Theorem 7. *Let $a \in \mathbb{R}$, and let $f \in \mathcal{PAP}_0(\mathbb{J}_a, X)$. Define $F : \mathbb{J}_a \rightarrow X$ by $F(t) = \int_a^t f(u) du$. Then $F \in \mathcal{PAP}(\mathbb{J}_a, X)$ if and only if there is a vector $A \in X$ such that $F - A \in \mathcal{PAP}_0(\mathbb{J}_a, X)$.*

Proof. The sufficiency is obvious. Now we show the necessity.

Since $F \in \mathcal{PAP}(\mathbb{J}_a, X)$, $F = G|_{\mathbb{J}_a} + \Phi$, where $G \in \mathcal{AP}(\mathbb{R}, X)$ and $\Phi \in \mathcal{PAP}_0(\mathbb{J}_a, X)$. To show the necessity, we need to show that G is a constant vector in X .

If it is not, there are $t', t'' \in \mathbb{R}$ with $t' < t''$ such that

$$\|G(t') - G(t'')\| = \epsilon > 0.$$

Since $G \in \mathcal{AP}(\mathbb{R}, X)$, for any $\tau \in P(\epsilon/4)$,

$$\|G(t') - G(t' + \tau)\| < \epsilon/4 \quad \text{and} \quad \|G(t'') - G(t'' + \tau)\| < \epsilon/4.$$

Combining these three inequalities, we have

$$(1) \quad \|G(t' + \tau) - G(t'' + \tau)\| > \epsilon/2 \quad (\tau \in P(\epsilon/4)).$$

Φ is uniformly continuous on $\mathbb{J}_a$ since F and G are. Let $\delta > 0$ be such that $\|f\|\delta < \epsilon/8$ and

$$(2) \quad \|\Phi(t_1) - \Phi(t_2)\| < \epsilon/16 \quad (t_1, t_2 \in \mathbb{J}_a, |t_1 - t_2| < \delta).$$

Set

$$C_1 = \left\{ t \in \mathbb{J}_a : \|\Phi(t)\| \geq \min \left\{ \frac{\epsilon}{8(t'' - t')}, \frac{\epsilon}{16} \right\} \right\},$$

$$C_2 = \left\{ t \in \mathbb{J}_a : \|f(t)\| \geq \min \left\{ \frac{\epsilon}{8(t'' - t')}, \frac{\epsilon}{16} \right\} \right\},$$

and $C_\epsilon = C_1 \cup C_2$. It follows from Proposition 4 that C_ϵ is an ergodic zero subset in $\mathbb{J}_a$. By Lemma 6, there exist a $\tau_0 \in P(\epsilon/4)$ and $(u, v) \subset \mathbb{J}_a$ such that $[t', t''] + \tau_0 \subset (u, v)$ and $m((u, v) \cap C_\epsilon) < \delta$.

We claim that $\|\Phi(t' + \tau_0)\| < \epsilon/8$ and $\|\Phi(t'' + \tau_0)\| < \epsilon/8$. In fact, if $t' + \tau_0 \in (u, v) \setminus C_\epsilon$, then $\|\Phi(t' + \tau_0)\| < \epsilon/16$; if $t' + \tau_0 \in (u, v) \cap C_\epsilon$, then by (2)

$$\|\Phi(t' + \tau_0)\| \leq \|\Phi(t' + \tau_0) - \Phi(t)\| + \|\Phi(t)\| < \epsilon/8,$$

where $t \in (u, v) \setminus C_\epsilon$ is such that $|t - (t' + \tau_0)| < \delta$.

Similarly we can show that $\|\Phi(t'' + \tau_0)\| < \epsilon/8$.

Now

$$\begin{aligned}
& \|G(t' + \tau_0) - G(t'' + \tau_0)\| \\
& \leq \left\| \int_a^{t'+\tau_0} f(u) du - \int_a^{t''+\tau_0} f(u) du \right\| + \|\Phi(t' + \tau_0)\| + \|\Phi(t'' + \tau_0)\| \\
& < \left\| \int_{t'+\tau_0}^{t''+\tau_0} f(u) du \right\| + \frac{\epsilon}{4} \\
& = \int_{[t'+\tau_0, t''+\tau_0] \setminus C_\epsilon} \|f(u)\| du + \int_{[t'+\tau_0, t''+\tau_0] \cap C_\epsilon} \|f(u)\| du + \frac{\epsilon}{4} \\
& \leq |t'' - t'| \cdot \frac{\epsilon}{8(t'' - t')} + \|f\|\delta + \frac{\epsilon}{4} < \frac{\epsilon}{2},
\end{aligned}$$

which contradicts (1).

Denote by $\mathcal{E}_0(\mathbb{J}_a, X)$ the set of functions $f \in \mathcal{E}(\mathbb{J}_a, X)$ such that $f(t) \rightarrow 0$ as $t \rightarrow \infty$. A function $f \in \mathcal{E}(\mathbb{J}_a, X)$ is called asymptotically almost periodic if $f = g|_{\mathbb{J}_a} + \varphi$, where $g \in \mathcal{AAP}(\mathbb{R}, X)$ and $\varphi \in \mathcal{E}_0(\mathbb{J}_a, X)$ [6]. Denote by $\mathcal{AAP}(\mathbb{J}_a, X)$ all such functions f . Since $\mathcal{E}_0(\mathbb{J}_a, X) \subset \mathcal{PAP}_0(\mathbb{J}_a, X)$, $\mathcal{AAP}(\mathbb{J}_a, X) \subset \mathcal{PAP}(\mathbb{J}_a, X)$. We have

Corollary 8 [5, 4.2]. *Let $a \in \mathbb{R}$, let $f \in \mathcal{E}_0(\mathbb{J}_a, X)$, and let $F : \mathbb{J}_a \rightarrow X$ be defined by $F(t) = \int_a^t f(u) du$. Then $F \in \mathcal{AAP}(\mathbb{J}_a, X)$ if and only if $\lim_{t \rightarrow \infty} F(t)$ exists.*

Proof. Necessity. Since $F \in \mathcal{AAP}(\mathbb{J}_a, X)$, $F = G|_{\mathbb{J}_a} + \Phi'$, where $G \in \mathcal{AAP}(\mathbb{R}, X)$ and $\Phi' \in \mathcal{E}_0(\mathbb{J}_a, X)$. By Theorem 7, $F = A + \Phi$, where $A \in X$ and $\Phi \in \mathcal{PAP}_0(\mathbb{J}_a, X)$. The uniqueness of the decomposition implies that $G = A$ and $\Phi \in \mathcal{E}_0(\mathbb{J}_a, X)$. Therefore, $F(t) = \int_a^t f(u) du \rightarrow A$ as $t \rightarrow \infty$.

The sufficiency is easy to prove.

The following example of [5, 4.1] can also be used here to show that the bounded integral of a pseudo-almost periodic function may fail to be pseudo-almost periodic even in the numerical case.

Example 9. Consider the function $f : \mathbb{J}_1 \rightarrow \mathbb{R}$ defined by

$$f(t) = \left(\frac{1}{t}\right) \cos(\log t) \quad (t \in \mathbb{J}_1).$$

Since $f(t) \rightarrow 0$ as $t \rightarrow \infty$, $f(t) \in \mathcal{PAP}_0(\mathbb{J}_1)$. The corresponding indefinite integral

$$F(t) = \int_1^t f(u) du = \sin(\log t) \quad (t \in \mathbb{J}_1)$$

defines a bounded function on $\mathbb{J}_1$; however, $F \notin \mathcal{PAP}(\mathbb{J}_a)$. That is because

$$\frac{1}{r} \int_1^r |\sin(\log t)| dt \not\rightarrow 0$$

as $r \rightarrow \infty$ and neither does $\frac{1}{r} \int_1^r |\sin(\log t) - A| dt$ for any $A \in \mathbb{C}$. But, if $F \in \mathcal{PAP}(\mathbb{J}_a)$, F differs from some member of $\mathcal{PAP}_0(\mathbb{J}_a)$ by a constant (Theorem 7).

Let $f \in \mathcal{PAP}(\mathbb{J}_a, X)$, and define $F(t) = \int_a^t f(u) du$ for $t \in \mathbb{J}_a$. To show $F \in \mathcal{PAP}(\mathbb{J}_a, X)$, we need to treat two cases. Theorem 7 allows us to control

the ergodic perturbation of f . We utilize the work of Kadets [4] in treating the almost periodic part of f . Also, as in [5], we need the following lemmas. In the lemmas, for a subset B of X , $\overline{\text{ccco}}(B)$ denotes the closed convex circled hull of B .

Lemma 10. *Let $a \in \mathbb{R}$, and let $f \in \mathcal{PAP}(\mathbb{J}_a, X)$. Let g and φ be the almost periodic component and the ergodic perturbation of f , respectively. If $F(t) = \int_a^t f(u) du$ and $G_a(t) = \int_a^t g(u) du$ for $t \in \mathbb{J}_a$, then*

$$G_a(\mathbb{J}_a) \subset 2\overline{\text{ccco}}(F(\mathbb{J}_a)).$$

Proof. Suppose that the conclusion does not hold; we can find a $t_0 \in \mathbb{J}_a$ such that $G_a(t_0) \notin 2\overline{\text{ccco}}(F(\mathbb{J}_a))$. In this case, we point out that $\|\varphi\| \neq 0$; otherwise we will have the conclusion. Also, $t_0 \neq a$ because $G_a(a) = 0 \in 2\overline{\text{ccco}}(F(\mathbb{J}_a))$. By the Hahn-Banach theorem, there is an $x^* \in X^*$ such that

$$\min_{y \in 2\overline{\text{ccco}}(F(\mathbb{J}_a))} |x^*(G_a(t_0) - y)| = \epsilon > 0.$$

Let

$$0 < \delta = \min \left\{ \frac{\epsilon}{3\|x^*\|(t_0 - a)}; \frac{\epsilon}{3\|x^*\|\|\varphi\|} \right\}$$

and

$$C = \{t \in \mathbb{J}_a : \|\varphi(t)\| \geq \delta\}.$$

By Proposition 4, C is an ergodic zero subset of $\mathbb{J}_a$. For $\delta > 0$ and $g \in \mathcal{AP}(\mathbb{R}, X)$, let $P(\delta)$ be the relatively dense subset of $\mathbb{R}$. By Lemma 6, there are $(u, v) \subset \mathbb{J}_a$ and a $\tau \in P(\delta)$ such that $[a, t_0] + \tau \subset (u, v)$ and $m((u, v) \cap C) < \delta$. Now,

$$\begin{aligned} \epsilon &= \min_{y \in 2\overline{\text{ccco}}(F(\mathbb{J}_a))} |x^*(G_a(t_0) - y)| \\ &\leq |x^*\{G_a(t_0) - [F(t_0 + \tau) - F(a + \tau)]\}| \\ &\leq \left| x^* \left\{ G_a(t_0) - \int_a^{t_0} g(u + \tau) du \right\} \right| \\ &\quad + \left| x^* \left\{ \int_a^{t_0} g(u + \tau) du - \int_{a+\tau}^{t_0+\tau} f(u) du \right\} \right| \\ &\leq \|x^*\| \int_a^{t_0} \|g(u) - g(u + \tau)\| du + \left| x^* \left\{ \int_a^{t_0} \varphi(u + \tau) du \right\} \right| \\ &\leq \|x^*\|(t_0 - a)\delta + \int_a^{t_0} |x^*\{\varphi(u + \tau)\}| du \\ &= \|x^*\|(t_0 - a)\delta + \int_{[a+\tau, t_0+\tau] \setminus C} |x^*\{\varphi(u)\}| du + \int_{[a+\tau, t_0+\tau] \cap C} |x^*\{\varphi(u)\}| du \\ &< \|x^*\|(t_0 - a)\delta + \delta\|x^*\|(t_0 - a) + \delta\|\varphi\|\|x^*\| < \epsilon, \end{aligned}$$

a contradiction.

Lemma 11 [5, 4.5]. *Let $a \in \mathbb{R}$, and let $g \in \mathcal{AP}(\mathbb{R}, X)$. Put $G_a(t) = \int_a^t g(u) du$ for $t \in \mathbb{J}_a$, and set*

$$G(t) = \begin{cases} \int_0^t g(u) du, & t \in \mathbb{J}_0, \\ -\int_t^0 g(u) du, & t \in \mathbb{R} \setminus \mathbb{J}_0. \end{cases}$$

Then

$$G(\mathbb{R}) \subset G([0, |a|]) + 2\overline{\text{cco}}(G_a(\mathbb{J}_a)).$$

Taken together, Lemmas 10 and 11 yield the following result.

Lemma 12. *Let a , f , g , φ , F , and G_a be as in Lemma 10. If $F(\mathbb{J}_a)$ is bounded [weakly relatively compact] {relatively compact} in X , then the same is true for $G(\mathbb{R})$.*

Let $\mathcal{B}(\mathbb{J}_a, X)$ denote the set of all bounded functions from $\mathbb{J}_a$ to X . Now we have the following theorem.

Theorem 13. *Let a , f , g , φ , and F be as in Lemma 10. Suppose either*

- (i) $F \in \mathcal{B}(\mathbb{J}_a, X)$ and X does not contain an isomorphic copy of c_0 , or
- (ii) $F(\mathbb{J}_a)$ is weakly relatively compact in X .

Then $F \in \mathcal{PAP}(\mathbb{J}_a, X)$ if and only if there is an $A \in X$ such that Φ , defined by

$$\Phi(x) = \int_0^x \varphi(u) du - A,$$

is in $\mathcal{PAP}_0(\mathbb{J}_a, X)$.

Proof. Necessity. We define the indefinite integrals $G_a : \mathbb{J}_a \rightarrow X$ and $G : \mathbb{R} \rightarrow X$ of g as in Lemma 11.

Since (i) or (ii) is satisfied, Lemma 12 shows that at least one of the following holds: $c_0 \not\subset X$, $G(\mathbb{R})$ is bounded, and $G(\mathbb{R})$ is weakly relatively compact in X . By [4, Theorem 1] in case (i), or by [4, Theorem 2] in case (ii), G is almost periodic. Therefore, $G_a = G|_{\mathbb{J}_a} - G(a) \in \mathcal{PAP}(\mathbb{J}_a, X)$, as is ψ , where

$$\psi(t) = \int_a^t \varphi(u) du = F(t) - \int_a^t g(u) du.$$

Now the necessity follows from Theorem 7.

The sufficiency is easy to prove; we omit the proof.

The following theorem was shown in [5, Theorem 4.11].

Theorem 14. *Let $f = g + \varphi$, where $g \in \mathcal{AP}(\mathbb{R}, X)$ and $\varphi \in \mathcal{E}_0(\mathbb{R}, X)$. Define $F : \mathbb{R} \rightarrow X$ by $F(t) = \int_0^t f(u) du$. Then F is in $\mathcal{WAP}(\mathbb{R}, X)$ if and only if either*

- (i) $F(\mathbb{R})$ is weakly relatively compact in X or
- (ii) $c_0 \not\subset X$ and $F \in \mathcal{B}(\mathbb{R}, X)$,

and the following limits exist and satisfy

$$(3) \quad \lim_{t \rightarrow \infty} \int_0^t \varphi(u) du = \lim_{t \rightarrow -\infty} \int_0^t \varphi(u) du.$$

Since $\mathcal{E}_0(\mathbb{R}, X) \subsetneq \mathcal{WAP}_0(\mathbb{R}, X)$, Ruess and Summers pointed out in [5, p. 33] that Theorem 14 does not answer the question: when is the integral of an $f \in \mathcal{WAP}(\mathbb{R}, X)$ again in $\mathcal{WAP}(\mathbb{R}, X)$?

The next theorem answers this question. Before stating the theorem, we show that if $f \in \mathcal{WAP}(\mathbb{R}, X)$ and $x^* \in X^*$, then $x^*f|_{\mathbb{J}_0} \in \mathcal{PAP}(\mathbb{J}_0)$, where $\mathbb{J}_0 = [0, \infty)$. In fact, $f = g + \varphi$, where $g \in \mathcal{AP}(\mathbb{R}, X)$ and $\varphi \in \mathcal{WAP}_0(\mathbb{R}, X)$ [3, Theorem 4.11]. $x^*g \in \mathcal{AP}(\mathbb{R})$, and $x^*\varphi \in \mathcal{WAP}_0(\mathbb{R})$. Since $\mathcal{WAP}_0(\mathbb{R}) \subset \mathcal{PAP}_0(\mathbb{R})$ [1, 4.3.13], $x^*\varphi \in \mathcal{PAP}_0(\mathbb{R})$. So $x^*\varphi|_{\mathbb{J}_0} \in \mathcal{PAP}_0(\mathbb{J}_0)$ and $x^*f|_{\mathbb{J}_0} \in \mathcal{PAP}(\mathbb{J}_0)$.

Theorem 15. Let $f \in \mathcal{WAP}(\mathbb{R}, X)$. Define $F: \mathbb{R} \rightarrow X$ by $F(t) = \int_0^t f(u) du$. Then F is in $\mathcal{WAP}(\mathbb{R}, X)$ if and only if either

- (i) $F(\mathbb{R})$ is weakly relatively compact in X , or
- (ii) $c_0 \not\subset X$ and $F \in \mathcal{B}(\mathbb{R}, X)$,

and there is a vector $A \in X$ such that

$$(4) \quad \psi - A \in \mathcal{WAP}_0(\mathbb{R}, X),$$

where $\psi: \mathbb{R} \rightarrow X$ is defined by $\psi(t) = \int_0^t \phi(u) du$.

Proof. Set $G(t) = \int_0^t g(u) du$ for $t \in \mathbb{R}$. With a proof similar to that for Lemma 12, we can show that if $F(\mathbb{R})$ is bounded (weakly relatively compact) in X , then the same is true for $G(\mathbb{R})$.

Necessity. Since $F \in \mathcal{WAP}(\mathbb{R}, X)$, $F(\mathbb{R})$ is weakly relatively compact, as is $G(\mathbb{R})$. So $G \in \mathcal{AP}(\mathbb{R}, X)$ [4, Theorem 2].

Now we show that (4) holds for some $A \in X$. Since $\psi \in \mathcal{WAP}(\mathbb{R}, X)$,

$$(5) \quad \psi = G_1 + \Phi_1,$$

where $G_1 \in \mathcal{AP}(\mathbb{R}, X)$ and $\Phi_1 \in \mathcal{WAP}_0(\mathbb{R}, X)$. We claim that G_1 is a constant function. For, suppose that there are $t_1, t_2 \in \mathbb{R}$ such that $G_1(t_1) \neq G_1(t_2)$. The almost periodicity of G_1 makes it possible to assume that t_1 and t_2 are in $\mathbb{J}_0$. Then we can find a $x^* \in X^*$ such that $x^*G_1(t_1) \neq x^*G_1(t_2)$. Since $\psi \in \mathcal{WAP}(\mathbb{R}, X)$, $x^*\psi|_{\mathbb{J}_0} \in \mathcal{PAP}(\mathbb{J}_0)$. By Theorem 7 there are an $A_{x^*} \in \mathbb{C}$ and a $\Phi_{x^*} \in \mathcal{PAP}_0(\mathbb{J}_0)$ such that

$$(6) \quad x^*\psi(t) = \int_0^t x^*\phi(u) du = A_{x^*} + \Phi_{x^*}(t) \quad (t \in \mathbb{J}_0).$$

Comparing (6) with (5) and using the uniqueness of the decomposition, we conclude that x^*G_1 is a constant function, a contradiction.

Sufficiency. It is obvious that $\psi \in \mathcal{WAP}(\mathbb{R}, X)$ if (4) holds for some $A \in X$. By the first paragraph in the proof and [4, Theorems 1 and 2], either (i) or (ii) is a sufficient condition for G to be in $\mathcal{AP}(\mathbb{R}, X)$, so $F \in \mathcal{WAP}(\mathbb{R}, X)$.

Remark 16. Theorem 14 is a corollary of Theorem 15. In fact, in Theorem 14, $A = \lim_{t \rightarrow \infty} \int_0^t \phi(u) du$.

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