A CHARACTERIZATION OF THE EXPONENTIAL DISTRIBUTION INVOLVING ABSOLUTE DIFFERENCES OF I.I.D. RANDOM VARIABLES

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(Communicated by Wei Y. Loh)

ABSTRACT. A probability measure μ on $[0, \infty)$ is said to have property H if for independent random variables X_1 and X_2 distributed according to μ the absolute difference $|X_1 - X_2|$ has the same distribution. Continuing previous work of Puri and Rubin we characterize the set of all distributions having property H .

1. INTRODUCTION

Let X , X_1 , and X_2 be independent random variables with values in $[0, \infty)$ having the common distribution μ . Then μ is said to have property H if X and $|X_1 - X_2|$ have the same distribution. In this paper we give a complete list of all distributions having property H .

The problem was already treated by Puri and Rubin [4]. They introduced property H and showed the following results on possible distributions μ :

- (a) If μ has a density, μ is an exponential distribution.
- (b) If X is bounded, then either $\mu(\{0\}) = 1$ or $\mu(\{0\}) = \mu(\{a\}) = \frac{1}{2}$ for some $a > 0$.
- (c) If X is discrete and unbounded, and the support of μ is not everywhere dense in $(0, \infty)$, then

$$\mu(\{0\}) = p_0, \quad \mu(\{k\tau\}) = 2p_0(1 - p_0)(1 - 2p_0)^{k-1}, \quad k = 1, 2, \dots,$$

for some $\tau > 0$ and $p_0 \in (0, \frac{1}{2})$.

However, the question whether there are other discrete distributions (with dense support) or singular distributions enjoying property H was still open. Using Fourier analytical methods (and some elementary complex analysis) we will prove that the list of distributions given in (a)–(c) is exhaustive. Thus, the exponential distribution is characterized by property H among all nonlattice distributions. This adds another characterization of the exponential law to the voluminous literature (see, e.g., Galambos and Kotz [4]).

Feller [2, pp. 208–209] considered a related problem of forming a long transmission line by welding together a large number of pieces of cable, assuming

Received by the editors August 12, 1992.

1991 *Mathematics Subject Classification*. Primary 62E10.

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0002-9939/94 \$1.00 + \$.25 per page

that the absolute deviations $Y_1, Y_2, \dots$ of these cables from the ideal value are i.i.d., absolutely continuous, nonnegative random variables, that these errors are additive and change sign if the corresponding piece of cable is reversed. Then attaching, for every $n \in \mathbb{N}$, the n th cable to the line of the $n-1$ previous ones such the n th error has a sign opposite to the already accumulated error, one obtains a Markov chain

$$U_1 = Y_1, \quad U_n = |U_{n-1} - Y_n|, \quad n \geq 2.$$

Feller shows that the stationary distribution of U_n has the density $x \rightarrow P(Y_1 > x)/E(Y_1)$ (see also Puri and Rubin [5] for the case when ν is assumed to have an absolutely continuous component).

Instead of attaching the cables piece-by-piece let us now form pairs of these pieces welded together such that the errors have opposite signs. Then put together these pairs to new pairs following the same rule, and so on. The distribution ν_2 of the first pairs is that of $|Y_1 - Y_2|$, and, inductively, the accumulated errors at the n th stage have the distribution ν_n of $|Y^{(n-1)} - \tilde{Y}^{(n-1)}|$, where $Y^{(n-1)}$ and $\tilde{Y}^{(n-1)}$ are independent with the common distribution ν_{n-1} . Clearly, if ν_n converges weakly, its limit possesses property H . It would be interesting to have conditions on the distribution of Y_1 specifying the type of the limiting distribution (among the possible ones given in (a)-(c)).

2. PRELIMINARIES

Let μ be a probability measure on $[0, \infty)$ having property H . Since the lattice case has been treated completely by Puri and Rubin [5], we assume that μ is not concentrated on a lattice.

Let $\varphi(t) = \int_0^\infty e^{itx} d\mu(x)$ be the characteristic function of μ . Then property H implies that

$$(1) \quad \varphi(t) = E(e^{it|X_1 - X_2|}) = (\mu \times \mu)(\Delta) + 2 \int_0^\infty \int_{[0, x)} e^{it(x-y)} d\mu(y) d\mu(x),$$

where $\Delta = \{(x_1, x_2) \in \mathbb{R}_+^2 \mid x_1 = x_2\}$, and

$$(2) \quad \varphi(-t) = (\mu \times \mu)(\Delta) + 2 \int_0^\infty \int_{(x, \infty)} e^{it(x-y)} d\mu(y) d\mu(x).$$

Obviously, (1) and (2) yield

$$(3) \quad \varphi(t) + \varphi(-t) = 2\varphi(-t)\varphi(t),$$

or $\operatorname{Re} \varphi(t) = |\varphi(t)|^2$. (3) will turn out to be the crucial relation in our approach. For complex z let

$$(4) \quad \varphi_1(z) = \int_0^\infty e^{izx} d\mu(x), \quad \varphi_2(z) = \int_0^\infty e^{-izx} d\mu(x).$$

Clearly, $\varphi_1(z)$ is analytic in $\{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ and continuous in $\{z \mid \operatorname{Im} z \geq 0\}$. Similarly, $\varphi_2(z)$ is analytic in $\{z \mid \operatorname{Im} z < 0\}$ and continuous in $\{z \mid \operatorname{Im} z \leq 0\}$. Moreover, φ_1 and φ_2 have meromorphic continuations on $\mathbb{C}$, because by (3) we can set

$$(5) \quad \varphi_1(z) = \varphi_2(z)/(2\varphi_2(z) - 1), \quad \operatorname{Im} z \leq 0;$$

$$(6) \quad \varphi_2(z) = \varphi_1(z)/(2\varphi_1(z) - 1), \quad \operatorname{Im} z \geq 0.$$

The function $t \rightarrow \varphi_2(it)$, $t \in (-\infty, 0]$, is strictly increasing. Thus the equation

$$(7) \quad \varphi_2(it) = \frac{1}{2}$$

has at most one real root in $(-\infty, 0)$.

Lemma 1. *If (7) has no root in $(-\infty, 0)$, $\varphi_1(z)$, as defined by (4) and (5), is an entire function.*

Proof. Since $\varphi_2(0) = 1$, the function φ_1 is analytic at 0. By a well-known result on characteristic functions, $\varphi_1(z)$ is analytic in $\{z \mid \operatorname{Im} z > -\alpha\}$ for some $\alpha > 0$ and has the representation

$$\varphi_1(z) = \int_0^\infty e^{izx} d\mu(x), \quad \operatorname{Im} z > -\alpha$$

(Lukacs [5, p. 193, Theorem 7.1.1]). Let α_0 be the supremum of all $\alpha > 0$ with the above property. We will show that $\alpha_0 = \infty$. Assume on the contrary that $\alpha_0 < \infty$. Since

$$\begin{aligned} \int_0^\infty e^{\alpha_0 x} d\mu(x) &= \lim_{0 < \alpha \uparrow \alpha_0} \int_0^\infty e^{\alpha x} d\mu(x) = \lim_{0 < \alpha \uparrow \alpha_0} \varphi_1(-i\alpha) \\ &= \lim_{0 < \alpha \uparrow \alpha_0} \frac{\varphi_2(-i\alpha)}{2\varphi_2(-i\alpha) - 1} < \infty, \end{aligned}$$

the function

$$(8) \quad F_{\alpha_0}(x) = \int_0^x e^{\alpha_0 u} d\mu(u), \quad x \geq 0,$$

is a distribution function belonging to some finite measure on $[0, \infty)$. By (5), $\varphi_1(z)$ is holomorphic at $z = -i\alpha_0$. Thus, the characteristic function

$$(9) \quad f_{\alpha_0}(t) = \int_0^\infty e^{itx} dF_{\alpha_0}(x) = \varphi_1(t - i\alpha_0)$$

of F_{α_0} is analytic at $t = 0$. By the result cited above, f_{α_0} has an analytic continuation to $\{z \mid \operatorname{Im} z > -\varepsilon\}$ for some $\varepsilon > 0$. Therefore, $\varphi_1(z)$ is analytic in $\{z \mid \operatorname{Im} z > -\alpha_0 - \varepsilon\}$. This contradicts the definition of α_0 .

Lemma 2. *Assume that (7) has the root $-t_0 \in (-\infty, 0)$. Then φ_1 is analytic in $\mathbb{C} \setminus \{-it_0\}$ and has a pole at $-it_0$.*

Proof. As in the derivation of Lemma 1 it is shown that φ_1 is analytic in $\{z \mid \operatorname{Im} z > -t_0\}$. If z has an imaginary part less than $-t_0$, we have $\varphi_2(i \operatorname{Im} z) < \frac{1}{2}$ and consequently

$$(10) \quad |\varphi_2(z)| \leq \int_0^\infty \exp\{x \operatorname{Im} z\} d\mu(x) = \varphi_2(i \operatorname{Im} z) < \frac{1}{2}.$$

From (5) and (10) we can conclude that φ_1 is analytic at z .

The same argument shows that φ_1 is analytic at any z for which $\operatorname{Im} z = -t_0$ and $\varphi_2(z) \neq \frac{1}{2}$. But if $z_1 \neq -it_0$ satisfies $\operatorname{Im} z_1 = -t_0$ and $\varphi_2(z_1) = \frac{1}{2}$, the function f_{t_0} (as defined in (9)) satisfies

$$f_{t_0}(\operatorname{Re} z_1) = \frac{1}{2} = f_{t_0}(0).$$

Since f_{t_0} is the characteristic function of F_{t_0} and $\operatorname{Re} z_1 \neq 0$, it follows that F_{t_0} belongs to a finite measure concentrated on a lattice; but then μ is also a

lattice distribution, a case which we have excluded. Hence, φ_1 is analytic in $\mathbb{C} \setminus \{-it_0\}$. By (5), $-it_0$ is a pole of φ_1 .

Lemma 3. *If f is the characteristic function of a probability measure ν on $\mathbb{R}$ which is not a point mass, we have*

$$(11) \quad \liminf_{t \rightarrow \infty} |f(t)| = \liminf_{t \rightarrow -\infty} |f(t)| < 1.$$

Proof. The equality in (11) follows from $f(-t) = \overline{f(t)}$. If U and V are independent random variables with common distribution ν , $Z = U - V$ has the characteristic function $g(t) = |f(t)|^2$. Assume that (11) does not hold. Then $g(t) \rightarrow 1$, as $t \rightarrow \infty$, and thus $E(\exp\{itnZ\}) = g(nt) \rightarrow 1$, as $n \rightarrow \infty$, for every $t \in \mathbb{R}$. By Lévy's continuity theorem, it follows that $nZ \xrightarrow{D} 0$, as $n \rightarrow \infty$. But this yields $Z \equiv 0$ with probability 1, which in turn implies that U and V must be constant.

Lemma 4. *The polynomial*

$$(12) \quad q_k(x) = \sum_{j=0}^{k-1} \frac{x^j}{j!(j+1)!(k-j-1)!}, \quad x \in \mathbb{R},$$

has exactly $k-1$ distinct negative roots.

Proof. We will determine the number of distinct roots of q_k by finding its standard Sturm sequence (see, e.g., Grove [3, p. 258]). For any $T > 0$ the number of roots of q_k in $[-T, 0]$ is equal to $V_{-T}(q_k) - V_0(q_k)$, where $V_a(q_k)$ is the number of sign changes of the sequence $g_0(a), \dots, g_m(a)$, defined as follows. The polynomials $g_0(x), \dots, g_m(x)$ (and further auxiliary polynomials $h_1(x), \dots, h_m(x)$) are given recursively by $g_0 = q_k$, $g_1 = q'_k$, and for $i \geq 1$ by

$$g_{i-1} = h_i g_i - g_{i+1}, \quad \text{where } \deg g_i > \deg g_{i+1},$$

and finally $g_{m-1} = h_m g_m$. Let

$$(13) \quad r_k(x) = \sum_{j=0}^k \frac{u^j}{j!(j+2)!(k-j)!}.$$

Then a straightforward comparison of coefficients yields the identities

$$(14) \quad g_1 = q'_k = r_{k-2}$$

$$(15) \quad g_0(x) = q_k(x) = \frac{x+k}{k-1} r_{k-2}(x) - \frac{1}{k-1} r_{k-3}(x),$$

$$(16) \quad r_j(x) = \frac{x+2j+1}{j(j+2)} r_{j-1}(x) - \frac{1}{j(j+2)} r_{j-2}(x),$$

$$(17) \quad r_0(x) = \frac{1}{2}.$$

It follows from (15) that $g_2(x) = c_{k-3} r_{k-3}(x)$ and then from (16) and (17) that $g_j(x) = c_{k-j-1} r_{k-j-1}(x)$ for $j = 3, \dots, k-1$ and that $m = k-1$, where the c_{k-j-1} are positive constants. In particular, $g_{k-j}(0) > 0$ and $(-1)^j g_{k-j}(-T) > 0$ for $j = 1, \dots, k$, if T is sufficiently large. The lemma is proved.

3. NONLATTICE DISTRIBUTIONS WITH PROPERTY H

We have to distinguish two cases.

Theorem 1. *If μ has property H and is not concentrated on a lattice and (7) has a solution $t_0 \in (-\infty, 0)$, then μ is an exponential distribution.*

Proof. We will show first that $\varphi_1(z)$ is bounded on $\{z \in \mathbb{C} \mid |z + it_0| > \eta\}$ for some $\eta > 0$. If $\text{Im } z < -t_0 - \varepsilon$, $\varepsilon > 0$, it follows that

$$|\varphi_2(z)| \leq \int_0^\infty e^{-(t_0+\varepsilon)x} d\mu(x) = C(\varepsilon) < \frac{1}{2},$$

so that

$$(18) \quad |\varphi_1(z)| \leq \frac{\frac{1}{2}}{|2C(\varepsilon) - 1|} = c(\varepsilon) < \infty.$$

If $\text{Im } z > t_0 + \varepsilon$,

$$(19) \quad |\varphi_1(z)| \leq \int_0^\infty e^{(t_0-\varepsilon)x} d\mu(x) = \tilde{c}(\varepsilon) < \infty.$$

The function $t \rightarrow \varphi_2(t - it_0)/\varphi_2(-it_0) = 2\varphi_2(t - it_0)$ is the characteristic function of a nonconstant random variable so that by Lemma 3

$$\liminf_{t \rightarrow -\infty} 2|\varphi_2(t - it_0)| = \liminf_{t \rightarrow -\infty} 2|\varphi_2(t - it_0)| < 1 - \delta$$

for some $\delta > 0$. One can therefore find sequences $0 < t_1^{(1)} < t_2^{(1)} < \dots \uparrow \infty$ and $0 > t_1^{(2)} > t_2^{(2)} > \dots \downarrow -\infty$ such that

$$(20) \quad |\varphi_2(t_j^{(l)} - it_0)| < \frac{1}{2}(1 - \delta) \quad \text{for all } j \geq 1 \text{ and } l = 1, 2.$$

Next note that

$$(21) \quad \left| \frac{d}{dz} \varphi_2(z) \right| = \left| \int_0^\infty x e^{-izx} d\mu(x) \right| \leq \int_0^\infty e^{-(t_0-\varepsilon)x} d\mu(x) = K(\varepsilon) < \infty \quad \text{on } \{z \mid -t_0 - \varepsilon \leq \text{Im } z \leq -t_0 + \varepsilon\}.$$

If $\varepsilon > 0$ is chosen sufficiently small, (20) and (21) together yield on $\{z \mid \text{Re } z = t_j^{(l)}, |\text{Im } z + it_0| \leq \varepsilon\}$ the estimate

$$\begin{aligned} |\varphi_2(z - it_0)| &\leq |\varphi_2(t_j^{(l)} - it_0)| + K(\varepsilon)|\varphi_2(z - it_0) - \varphi_2(t_j^{(l)} - it_0)| \\ &< \frac{1}{2}(1 - \delta) + K(\varepsilon)\varepsilon. \end{aligned}$$

Since $K(\varepsilon)$ remains bounded, as $\varepsilon \rightarrow 0$, it follows that for sufficiently small $\varepsilon > 0$ we have

$$\varphi_1(t_j^{(l)} + is) = \frac{|\varphi_2(t_j^{(l)} + is)|}{|2\varphi_2(t_j^{(l)} + is) - 1|} \leq \frac{1 - \delta}{2\delta}, \quad s \in [-t_0 - \varepsilon, -t_0 + \varepsilon].$$

Therefore, $|\varphi_1(z)|$ is bounded by $\max[c(\varepsilon), \tilde{c}(\varepsilon), (1 - \delta)/2\delta]$ on the boundary of the rectangle having the four edges $t_j^{(l)} - i(t_0 - \varepsilon)$, $t_j^{(l)} - i(t_0 + \varepsilon)$, $t_{j+1}^{(l)} - i(t_0 - \varepsilon)$, $t_{j+1}^{(l)} - i(t_0 + \varepsilon)$, $l = 1, 2$. By the maximum principle for analytic functions, $|\varphi_1(z)|$ is also bounded by the same constant inside each of these rectangles. Recalling also (18) and (19) we see that $|\varphi_1(z)|$ is bounded on any set of the form $\{z \mid |z + it_0| > \eta\}$, where $\eta > 0$.

It is now an easy consequence of Liouville's theorem that φ_1 can be written in the form

$$(22) \quad \varphi_1(z) = p(z)/(z + it_0)^n, \quad z \neq -it_0,$$

where $p(z)$ is a polynomial of degree $m \leq n$ satisfying $p(-it_0) \neq 0$. Using the basic equation (3) we find that

$$(23) \quad \frac{p(t)}{(t + it_0)^n} + \frac{p(-t)}{(-t + it_0)^n} = 2(-1)^n \frac{p(t)p(-t)}{(t^2 + t_0^2)^n}, \quad t \in \mathbb{R}.$$

We introduce the auxiliary polynomial $P(z) = (it_0)^{-n}p(it_0z)$, $z \in \mathbb{C}$. If z is purely imaginary, (23) shows that

$$(24) \quad \frac{P(z)}{(z + 1)^n} + \frac{P(-z)}{(1 - z)^n} = \frac{2P(z)P(-z)}{(z + 1)^n(1 - z)^n}.$$

Obviously, (24) must even hold for all $z \in \mathbb{C} \setminus \{1, -1\}$. It follows from (24) that

$$\frac{P(-z)(1 + z)^n}{P(z)} = 2P(-z) - (1 - z)^n.$$

Thus the polynomial $P(-z)(1 + z)^n$ is divisible by $P(z)$.

Since $P(-1) = p(-it_0) \neq 0$, $P(-z)$ must be divisible by $P(z)$. This entails $P(z) = (-1)^m P(-z)$. Thus (24) yields $P(z) = \frac{1}{2}[(-1)^m(1 - z)^n + (1 + z)^n]$. Since $P(0) = (it_0)^{-n}p(0) = \varphi_1(0) = \varphi(0) = 1$, m must be even so that

$$(25) \quad P(z) = \frac{1}{2}[(1 - z)^n + (1 + z)^n].$$

From (22) and (25) we obtain

$$(26) \quad \varphi(t) = \frac{(it_0 - t)^n + (t + it_0)^n}{2(t + it_0)^n}.$$

For $n = 1$ we have $\varphi(t) = it_0/(t + it_0)$, which is the characteristic function of the exponential distribution with mean $1/t_0$. If n is even, the right-hand side of (26) converges to 1, as $t \rightarrow \infty$, which is impossible, since $\liminf_{t \rightarrow \infty} |\varphi(t)| < 1$.

It remains to show that for odd $n \geq 3$ the function φ in (26) is not a characteristic function. Assume the contrary. We can then apply Fourier inversion and the residue theorem to determine the density $\psi(x)$ associated with φ :

$$(27) \quad \begin{aligned} \psi(x) &= (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-itx} \varphi(t) dt = -(2\pi)^{-1} 2\pi \operatorname{Res}(e^{-izx} \varphi(z); -it_0) \\ &= t_0 e^{-t_0 x} n! \sum_{j=0}^{n-1} \frac{(-1)^j (2t_0 x)^j}{j!(j+1)!(n-j-1)!} \\ &= t_0 e^{-t_0 x} n! q_n(-2t_0 x), \quad x > 0, \end{aligned}$$

where q_n is defined by (13) (note that the Fourier integral in (27) exists because $(it_0 - t)^n + (t + it_0)^n$ has degree $n - 1$; for the second equation see, e.g., Behnke and Sommer [1, p. 217, formula 13]). By Lemma 4, q_n has simple negative roots, and this implies that ψ is negative on some interval in $(0, \infty)$. Therefore, ψ is not a density if $n \geq 2$. The theorem is proved.

Theorem 2. Let $\varphi_2(it) > \frac{1}{2}$ for all $t \in (-\infty, 0)$. Then $\mu(\{0\}) = 1$ or $\mu(\{0\}) = \mu(\{a\}) = \frac{1}{2}$ for some $a > 0$.

Proof. By Lemma 1, φ_1 is an entire function. Since

$$\int_0^\infty e^{tx} d\mu(x) = \varphi_2(it) > \frac{1}{2} \quad \text{for every } t < 0,$$

we must have $\beta = \mu(\{0\}) \geq \frac{1}{2}$. If $\beta > \frac{1}{2}$, we have

$$\begin{aligned} \lim_{t \rightarrow \infty} \varphi_1(it) &= \lim_{t \rightarrow \infty} \int_0^\infty e^{-tx} d\mu(x) = \beta, \\ \lim_{t \rightarrow -\infty} \varphi_1(it) &= \lim_{t \rightarrow -\infty} \varphi_2(it)/(2\varphi_2(it) - 1) = \beta/(2\beta - 1). \end{aligned}$$

As $|\varphi_1(z)| \leq \varphi_1(i \operatorname{Im} z)$, $\varphi_1(z)$ is a bounded entire function and thus constant: $\varphi_1(z) \equiv \varphi_1(0) = 1$. Hence $\mu = \delta_0$.

If $\beta = \frac{1}{2}$, we have

$$\frac{1}{2} = P(|X_1 - X_2| = 0) = \int_0^\infty \mu(\{x\}) d\mu(x) = \frac{1}{4} + \sum_j \mu(\{x_j\})^2,$$

where the x_j are the positive atoms of μ . But the relations $\sum_j \mu(\{x_j\})^2 = \frac{1}{4}$ and $\sum_j \mu(\{x_j\}) \leq \frac{1}{2}$ together imply that there is only one positive atom.

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