ON THE MÜNTZ RATIONAL APPROXIMATION RATE

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ABSTRACT. The present paper constructs a counterexample to show that a conjecture of Newman concerning rational approximation rate of arbitrary Markov system is generally not true.

1

Let $C_{[0, b]}$ be the class of all real continuous functions f on $[0, b]$.¹ For $f \in C_{[0, b]}$,

$$\omega(f, t) = \max\{|f(x+h) - f(x)| : x \in [0, b-h], 0 < h \leq t\},$$

$$\|f\|_{[0, b]} = \max_{x \in [0, b]} |f(x)|, \quad \text{and} \quad \|f\| = \|f\|_{[0, 1]}.$$

Given a subspace S of $C_{[0, b]}$, let

$$R(S) = \{P(x)/Q(x) : P(x) \in S, Q(x) \in S, Q(x) > 0, x \in (0, b)\},$$

where we assume that $P(0)/Q(0) = \lim_{x \rightarrow 0+} P(x)/Q(x)$ is finite in the case $Q(0) = 0$. For a sequence $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$, write

$$R(\Lambda) = R(\text{span}\{x^{\lambda_n}\}).$$

From Müntz's theorem, it is well known that the linear combinations of $\{x^{\lambda_n}\}$ for

$$(1) \quad 0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

are dense if and only if $\sum_{n=1}^{\infty} 1/\lambda_n = \infty$.

Concerning the rational case, in 1976, Somorjai [5] showed a beautiful result that, under (1), $R(\Lambda)$ is always dense in $C_{[0, 1]}$. In 1978, Bak and Newman [2] proved that, if $\{\lambda_n\}$ is any sequence of distinct positive real numbers, then $R(\Lambda)$ is dense in $C_{[0, 1]}$, too. Our recent work [6] showed that $R(\Lambda)$ is always dense for any sequence of real numbers $\{\lambda_n\}$ with infinitely many distinct elements.

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¹For $b = \infty$, we assume further that $\lim_{x \rightarrow \infty} f(x)$ exists and is finite.

Write

$$\begin{aligned}\Lambda_n &= \{\lambda_1, \lambda_2, \dots, \lambda_n\}, \\ E_n &= \{e^{-\lambda_1 t}, e^{-\lambda_2 t}, \dots, e^{-\lambda_n t}\}, \quad E = \{e^{-\lambda_1 t}, e^{-\lambda_2 t}, \dots\}, \\ R(\Lambda_n) &= R(\text{span}\{x^{\lambda_k}\}: \lambda_k \in \Lambda_n), \quad R(E_n) = R(\text{span}(E_n)), \\ R_n(f, \Lambda) &= \min_{\gamma \in R(\Lambda_n)} \|f - \gamma\| \quad \text{for } f \in C_{[0, 1]}, \\ R_n(f, E) &= \min_{\gamma \in R(E_n)} \|f - \gamma\|_{[0, \infty)} \quad \text{for } f \in C_{[0, \infty)}.\end{aligned}$$

On the quantitative Müntz rational approximation rate, Bak [1] proved that, if $f \in C_{[0, 1]}$ and $\{\lambda_n\}$ is a sequence with (1) and $\lambda_k - \lambda_{k-1} \geq k$ for all $k \geq 2$, then

$$R_n(f, \Lambda) \leq C\omega(f, n^{-1}),$$

where here and throughout the paper C always indicates an absolute constant which may have different values in different places, while $C(x)$ indicates a positive constant only depending upon x .

Newman [4] raised the following two problems on this topic (Newman said that even he did not believe Problem 10.4):

Problem 10.3. Is it true that for any $f \in C_{[0, 1]}$ there exists $R(x) \in R(\Lambda_n)$ such that

$$\|f - R\| \leq C\omega(f, n^{-1})?$$

Problem 10.4. The same conclusion as the above problem holds where x^{λ_k} are replaced by $\Psi_k(x)$ for any Markov system $\{\Psi_k(x)\}$.

An infinite Markov system on an interval $[a, b]$ is a collection of continuous functions on $[a, b]$, $\mathcal{M} := \{\Psi_1 = 1, \Psi_2, \Psi_3, \dots\}$, with the property that, if an element of the real linear span of the first n vanishes at n points, then it vanishes identically.

The present paper will construct a counterexample to show Problem 10.4 is generally not true for a Markov system in the unbounded interval $[0, \infty)$.

2

We will prove the following result, which gives a negative answer to Problem 10.4 (for a Markov system in $[0, \infty)$).

Theorem. Let $\Lambda^* = \{\lambda_k^*\}_{k=1}^\infty$ and

$$\begin{aligned}A = \{ & A_{1,1}, A_{1,2}, A_{1,3}, A_{1,4}, A_{2,1}, A_{2,2}, A_{2,3}, A_{2,4}, \dots, A_{1,2^n}, \\ & A_{2,2^n}, \dots, A_{n-1,2^n}, A_{n,1}, A_{n,2}, \dots, A_{n,2^n}, A_{1,2^{n+1}}, \dots, A_{n,2^{n+1}}, \\ & A_{1,2^{n+2}}, \dots, A_{n,2^{n+2}}, \dots, A_{1,2^{n+1}-1}, \dots, A_{n,2^{n+1}-1}, \dots\} := \{\lambda_k\}_{k=1}^\infty,\end{aligned}$$

where

$$A_{i,j} = \lambda_i^* + j - 1, \quad i, j = 1, 2, \dots, \quad \lambda_i^* = \begin{cases} 0, & i = 1, \\ i^{-2}, & i \geq 2. \end{cases}$$

Furthermore, let $\{s_n\}$ be an increasing sequence with the following properties:

$$(2) \quad \lim_{n \rightarrow \infty} s_n = +\infty, \quad s_n \sim s_{2n},^2 \quad \lim_{n \rightarrow \infty} (s_n/n) = 0.$$

Then E is a Markov system on $[0, \infty)$ and there exists a function $f \in C_{[0, \infty]}$ such that

$$\limsup_{n \rightarrow \infty} \frac{R_n(f, E)}{s_n \omega(f, n^{-1})} > 0.$$

Proof. It is a very clear fact that E is a Markov system since λ_k , $k = 1, 2, \dots$, are distinct. Let $T_n(t) := T_n(t, \Lambda^*)$ be the generalized Chebyshev polynomial of degree n associated with the Markov system $\{e^{-\lambda_1^* t}, e^{-\lambda_2^* t}, \dots\}$ on $[0, \infty)$, that is, the linear form

$$T_n(t) = C_0 \left(e^{-\lambda_n^* t} - \sum_{k=1}^{n-1} C_k e^{-\lambda_k^* t} \right),$$

where C_k , $k = 1, 2, \dots, n-1$, are chosen so that $\sum_{k=1}^{n-1} C_k e^{-\lambda_k^* t}$ is the best approximant to $e^{-\lambda_n^* t}$ from $\text{span}\{e^{-\lambda_k^* t}\}_{k=1}^{n-1}$ and where C_0 is chosen so that $\|T_n\|_{[0, \infty)} = 1$.

By the well-known results in approximation theory, there exists an ordinary polynomial $Q_n(x)$ with sufficiently large degree $m_n \geq 1$ such that

$$(3) \quad \|T_n(t) - Q_n(e^{-t})\|_{[0, \infty)} \leq n^{-1}.$$

Now we may select a sequence $\{n_l\}$ by induction. Let $n_1 = 2$. Suppose n_l is given. Let

$$(4) \quad M_l^* := n_l 2^{n_l},$$

$$(5) \quad M_l := 2M_l^* + 2.$$

Choose n_{l+1} satisfying the following properties:

$$(6) \quad n_{l+1} \geq [\log_2 m_{M_l} + 1],$$

$$(7) \quad \varepsilon_{l+1} := \sqrt{\frac{S_{M_{l+1}}}{M_{l+1}}} \leq \min \left\{ \frac{\varepsilon_l S_{M_l}^{-1}}{4}, \sum_{k=1}^l \varepsilon_k \left\| \frac{d}{dt} Q_l(e^{-t}) \right\|_{[0, \infty)} \right\}.$$

By (2), (7) is possible. Define

$$F_l(t) = \sum_{k=1}^l \varepsilon_k Q_{M_k}(e^{-t}), \quad f(t) = \sum_{k=1}^{\infty} \varepsilon_k Q_{M_k}(e^{-t}).$$

It is quite clear that $f \in C_{[0, \infty]}$ follows from (3) and (7). For any rational $r(t) \in R(E_{M_l^*})$,

$$(8) \quad \|f(t) - r(t)\|_{[0, \infty)} \geq \|F_l(t) - r(t)\|_{[0, \infty)} - 2 \sum_{k=l+1}^{\infty} \varepsilon_k.$$

² By $A_n \sim B_n$, we mean that there exists a positive constant M independent of n such that $M^{-1} \leq A_n/B_n \leq M$.

We have the estimate

$$(9) \quad \|F_l - r\|_{[0, \infty)} \geq \varepsilon_l(1 - M_l^{-1}).$$

In fact, that (9) fails will lead to a contradiction. Because of the definition, $T_{M_l}(t)$ oscillates between ± 1 exactly M_l times on $[0, \infty)$.³ Assume, for $0 \leq x_1 < x_2 < \cdots < x_{M_l} \leq \infty$,

$$T_{M_l}(x_j) = \varepsilon(-1)^j, \quad \varepsilon = \pm 1.$$

From (3),

$$\operatorname{sgn}(Q_{M_l}(x_j)) = \operatorname{sgn}(T_{M_l}(x_j)), \quad |Q_{M_l}(x_j)| \geq 1 - M_l^{-1}.$$

Suppose (9) fails. Then

$$\operatorname{sgn}((r - F_{l-1})(x_j)) = \operatorname{sgn}((\varepsilon_l Q_{M_l} - F_l + r)(x_j)) = \operatorname{sgn}(Q_{M_l}(x_j)),$$

which means $r - F_{l-1}$ vanishes at least $M_l - 1$ times on $[0, \infty)$. It is impossible since $r - F_{l-1} \in R(E_{M_l-2})$ by (4)–(6) and by some direct arguments.

Therefore, combining (7)–(9) yields that

$$\|f(t) - r(t)\|_{[0, \infty)} \geq \varepsilon_l(1 - M_l^{-1}) - \varepsilon_l s_{M_l}^{-1},$$

so that

$$(10) \quad R_{M_l^*}(f, E) \geq \varepsilon_l(1 - M_l^{-1}) - \varepsilon_l s_{M_l}^{-1}.$$

On the other hand,

$$\begin{aligned} \omega(f, (M_l^*)^{-1}) &\leq 3M_l^{-1} \sum_{k=1}^{l-1} \varepsilon_k \left\| \frac{d}{dt} Q_{M_k}(e^{-t}) \right\|_{[0, \infty)} + 3\varepsilon_l \omega(Q_{M_l}(e^{-t}), M_l^{-1}) \\ &\quad + 4 \sum_{k=l+1}^{\infty} \varepsilon_k := \Sigma_1 + \Sigma_2 + \Sigma_3. \end{aligned}$$

It follows from (7) that

$$\Sigma_3 \leq 2\varepsilon_l s_{M_l}^{-1} \quad \text{and} \quad \Sigma_1 \leq 3\varepsilon_l s_{M_l}^{-1}.$$

Applying (2) and an inequality for derivatives of generalized Müntz polynomials of Newman [3] we have

$$\Sigma_2 \leq 3\|Q_{M_l}(e^{-t}) - T_{M_l}(t)\|_{[0, \infty)} + 3M_l^{-1}\|T'_{M_l}\|_{[0, \infty)} = O(M_l^{-1})$$

since $\sum_{k=2}^{\infty} \lambda_k^* < +\infty$. Putting together the above estimates and (10) we then have

$$\frac{R_{M_l^*}(f, E)}{s_{M_l^*} \omega(f, (M_l^*)^{-1})} \geq \frac{CR_{M_l^*}(f, E)}{s_{M_l} \omega(f, (M_l^*)^{-1})} > C.$$

The theorem is completed. $\square$

³ We may take ∞ as an alternating point.

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