ON THE NUMBER OF GENERATORS OF MODULES OVER POLYNOMIAL RINGS

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ABSTRACT. In this paper we prove the following

Theorem. Let $B = A[X_1, \dots, X_n]$, where A is a universally catenary equidimensional ring. Let M be a finitely generated B -module of rank r . Denote by d the Krull dimension of A , by $\mu(M)$ the minimal number of generators of M , and by I_M the (radical) ideal which defines the set of primes of B at which M is not locally free. Assume that

$$\mu(M/I_M M) \leq \eta \quad \text{and} \quad \eta \geq \max\{d + r, \dim B/I_M + r + 1\},$$

where η is a positive integer. Then $\mu(M) \leq \eta$.

This improves a result of R. G. López, *On the number of generators of modules over polynomial affine rings*, Math. Z. **208** (1991), 11–21.

In [L] López proved the following theorem.

Theorem [L]. Let A be an affine domain, and put $B = A[X_1, \dots, X_n]$. Let M be a finitely generated B -module of rank r . Denote by d the Krull dimension of A , by $\mu(M)$ the minimal number of generators of M , and by I_M the (radical) ideal which defines the set of primes of B at which M is not locally free. Assume that

$$\mu(M/I_M M) \leq \eta \quad \text{and} \quad \eta \geq \max\{d + r, \dim B/I_M + r + 1\},$$

where η is a positive integer. Then $\mu(M) \leq \eta$.

We can drop the condition of A being an affine domain in the theorem and replace it with A being a universally catenary equidimensional ring. At the same time we can give a much shorter proof of the above result, i.e., we prove

Theorem. Let A be a universally catenary equidimensional ring and the rest as in the above theorem. The same conclusion holds.

The notation is the same as in [L]. For universally catenary equidimensional rings see [M].

Let us start the proof of the theorem by recalling [L, Proposition 2.1].

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Proposition 2.1 [L]. *Let A be a ring of dimension d , and put*

$$B = A[X_1, \dots, X_n].$$

Let M be a B -module of rank r , and denote by I_M the (radical) ideal which defines the set of primes of B at which M is not locally free. If

$$(1) \text{ ht}(I_M) \geq n + 1,$$

$$(2) \mu(M/I_M M) \leq \eta, \eta \geq \max\{3, d + r\},$$

then $\mu(M) \leq \eta$.

López's Corollary 2.2 also holds when A is a universally catenary equidimensional ring.

Corollary. *Let A be a universally catenary equidimensional ring of Krull dimension d , set $B = A[X_1, \dots, X_n]$, let M be a B -module of rank r , and denote by I_M the (radical) ideal which defines the set of primes of B at which M is not locally free. Assume that*

$$\mu(M/I_M M) \leq \eta \quad \text{and} \quad \eta \geq \max\{3, d + r, \dim B/I_M + r + 1\}.$$

Then $\mu(M) \leq \eta$.

Proof. This is almost the same proof as in [L] because the only property of affine domain used there was universally catenary equidimensionality.

Put $h = \text{ht}(I_M)$. If $h \geq n + 1$, then apply [L, Proposition 2.1]. If $h < n + 1$, put $t = n - h + 1$ and $A' = A[X_1, \dots, X_t]$. Then $B = A'[X_{t+1}, \dots, X_n]$, $\dim(A') = \dim(A) + t = d + n - h + 1$. Since A is universally catenary equidimensional, B is catenary equidimensional. Therefore, $\dim(B/I_M) = \dim(B) - \text{ht}(I_M) = d + n - h$. Now by assumption $\eta \geq \dim B/I_M + r + 1 = d + n - h = \dim(A') + r$, so again we can apply [L, Proposition 2.1] to A' , B , M , and I_M .

The remaining proof differs substantially from that of [L]. First we remark that η as given in the theorem ≥ 1 . If $\eta \geq 3$, then η satisfies the conditions set in the corollary and we are done. Therefore, we may assume that $\eta \leq 2$ from now on. There are only two possibilities; $\eta = 1$ or $\eta = 2$.

When $\eta = 1$, $r = 0$, so M is torsion and $I_M = \text{radical of } \text{Ann } M$. Hence, $\mu(M) = \mu(M/I_M M)$.

When $\eta = 2$, r could be either 0 or 1. If $r = 0$, M is torsion again and we are done, so the remaining case would be $r = 1$; this implies that $d \leq 1$. The theorem now follows from the following lemma which is a bit more general.

Lemma. *Let A be a noetherian ring of $\dim \leq 1$, $B = A[X_1, \dots, X_n]$. Let M be a finitely generated B -module of rank r and I_M the radical ideal of B which defines the set of primes of b at which M is not locally free. Assume $\mu(M/I_M M) \leq r + 1$. Then $\mu(M) \leq r + 1$.*

Proof. Put $S = 1 + I_M$. S is a multiplicatively closed subset of B , and $(I_M)_S$ is in the Jacobson radical of B_S . $\mu(M/I_M M) \leq r + 1$ implies that $\mu(M_S/(I_M)_S M_S) \leq r + 1$, so Nakayama's lemma shows $\mu(M_S) \leq r + 1$. As B is a noetherian ring and M is a finitely generated B -module, there exists an element $s \in S$, i.e., $b \in I_M$, $s = 1 + b$, such that M_s is $(r + 1)$ -generated. We can choose the generating set in M , say $\{m_1, \dots, m_{r+1}\}$ in M , to generate M_{1+b} as a B_{1+b} -module. Hence there is a surjective map $B_{1+b}^{\oplus r+1} \xrightarrow{\sigma} M_{1+b} \rightarrow 0$, where

the B_{1+b} -module homomorphism σ is defined as follows: if $\{e_1, \dots, e_{r+1}\}$ is a basis for the free B_{1+b} -module $B_{1+b}^{\oplus r+1}$, $\sigma(e_1) = m_1, \dots, \sigma(e_{r+1}) = m_{r+1}$. Let K be the kernel. We have the following exact sequence:

$$(1) \quad 0 \rightarrow K \rightarrow B_{1+b}^{\oplus r+1} \xrightarrow{\sigma} M_{1+b} \rightarrow 0.$$

Since $b \in I_M$, M_b is a projective module of rank r . We have another exact sequence

$$(2) \quad 0 \rightarrow \Lambda^r M_b^* \rightarrow M_b \oplus \Lambda^r M_b^* \xrightarrow{\pi} M_b \rightarrow 0,$$

where π is simply the projection on the first factor, $\Lambda^r M_b^*$ is the r th exterior power of M_b^* , and M_b^* is the dual of M_b , i.e., $M_b^* = \text{Hom}(M_b, B_b)$. Localize at b ; (1) becomes the exact sequence of modules over $B_{b(1+b)}$

$$(3) \quad 0 \rightarrow K_b \rightarrow B_{b(1+b)}^{\oplus r+1} \rightarrow M_{b(1+b)} \rightarrow 0.$$

Since $M_{b(1+b)}$ is projective, the exact sequence (3) splits, $B_{b(1+b)}^{\oplus r+1} \xrightarrow{\alpha} K_b \oplus M_{b(1+b)}$. Notice $\alpha(e_i) = (n_i, \sigma(e_i))$ for some $n_i \in K_b$. Taking the $(r+1)$ th exterior power of both sides we get $K_b \xrightarrow{\beta} (\Lambda^r M_b^*)_{1+b}$. Therefore, $B_{b(1+b)}^{\oplus r+1} \xrightarrow{\alpha} K_b \oplus M_{b(1+b)} \xrightarrow{(\beta, \text{Id})} \Lambda^r M_{b(1+b)}^* \oplus M_{b(1+b)}$, where Id is the identity map on the second factor $M_{b(1+b)}$. Denote the composition of the $B_{b(1+b)}$ -module isomorphism by γ , i.e., $\gamma = (\beta, \text{Id}) \circ \alpha$.

The following commutative square satisfies the Milnor Patching Property [Mi]:

$$(4) \quad \begin{array}{ccc} B & \longrightarrow & B_{b+1} \\ \downarrow & & \downarrow \\ B_b & \longrightarrow & B_{b(1+b)} \end{array}$$

We can glue the free module $B_{1+b}^{\oplus r+1}$ over B_{1+b} and the projective module $M_b \oplus \Lambda^r M_b^*$ over B_b , along with the isomorphism $\gamma : B_{b(1+b)}^{\oplus r+1} \simeq M_{b(1+b)} \oplus \Lambda^r M_{b(1+b)}^*$ to get a projective B -module P of rank $(r+1)$. Its top exterior power $\Lambda^{r+1} P$ is given by gluing B_{1+b} over B_{1+b} and $\Lambda^{r+1}(M_b \oplus \Lambda^r M_b^*)$ over B_b , along with the isomorphism $\Lambda^{r+1} \gamma : B_{b(1+b)}^{\Lambda^{r+1} \gamma} \simeq \Lambda^{r+1}(M_{b(1+b)} \oplus \Lambda^r M_{b(1+b)}^*)$. Since $\Lambda^{r+1}(M_b \oplus \Lambda^r M_b^*) \xrightarrow{\phi} B_b$ over B_b (there are many choices of ϕ here, but let us fix an isomorphism ϕ for the rest of the proof), $\Lambda^{r+1} P$ is isomorphic to a projective B -module L given by gluing B_{1+b} over B_{1+b} and B_b over B_b , along with the isomorphism

$$\psi = \phi_{1+b} \circ \Lambda^{r+1} \gamma : B_{b(1+b)}^{\Lambda^{r+1} \gamma} \simeq \Lambda^{r+1}(M_{b(1+b)} \oplus \Lambda^r M_{b(1+b)}^*) \xrightarrow{\phi_{1+b}} B_{b(1+b)},$$

where ϕ_{1+b} is just the localization of ϕ . ψ is given by a unit in $B_{b(1+b)}$.

$$(5) \quad \text{If } \psi = 1 \text{ in } B_{b(1+b)},$$

then L is isomorphic to B . Hence $\Lambda^{r+1}P \simeq B$. Then, by the theorem of Bhatwadekar and Roy [BR], $P \simeq B^{\oplus r} \oplus \Lambda^{r+1}P \simeq B^{\oplus r+1}$ since $\dim(A) \leq 1$ by assumption. Now the diagram

$$(6) \quad \begin{array}{ccc} B_{b(1+b)}^{\oplus r+1} & \xrightarrow{\sigma} & M_{b(1+b)} \\ \gamma \downarrow & & \parallel \\ M_{b(1+b)} \oplus \Lambda^r M_{b(1+b)}^* & \xrightarrow{\pi} & M_{b(1+b)} \end{array}$$

is commutative because $\pi \circ \gamma(e_i) = \pi \circ (\beta, \text{Id}) \circ \alpha(e_i) = m_i = \sigma(e_i)$. We can glue the horizontal maps to get a surjective map from P to M , so we are done. The goal of the rest of the proof would be to alter the isomorphism γ without violating the commutativity of the above diagram and to make (5) hold.

If $\psi \neq 1$, we compose γ with

$$\gamma_1 : M_{b(1+b)} \oplus \Lambda^r M_{b(1+b)}^* \xrightarrow{(\text{id}_{M_{b(1+b)}}, \psi^{-1})} M_{b(1+b)} \oplus \Lambda^r M_{b(1+b)}^*.$$

We get $\gamma' = \gamma_1 \circ \gamma : B_{b(1+b)}^{\oplus r+1} \simeq M_{b(1+b)} \oplus \Lambda^r M_{b(1+b)}^*$.

Glue $B_{b(1+b)}^{\oplus r+1}$ over $B_{b(1+b)}$ and $M_b \oplus \Lambda^r M_b^*$ over B_b along the isomorphism $\gamma' : B_{b(1+b)}^{\oplus r+1} \simeq M_{b(1+b)} \oplus \Lambda^r M_{b(1+b)}^*$ to get a projective B -module P' . Since we still have the commutative diagram

$$(7) \quad \begin{array}{ccc} B_{b(1+b)}^{\oplus r+1} & \xrightarrow{\sigma} & M_{b(1+b)} \\ \gamma' \downarrow & & \parallel \\ M_{b(1+b)} \oplus \Lambda^r M_{b(1+b)}^* & \xrightarrow{\pi} & M_{b(1+b)} \end{array}$$

we can glue the horizontal maps to get a surjective map from P' to M .

The top exterior power, $\Lambda^{r+1}P'$, now is given by gluing B_{1+b} over B_{1+b} and $\Lambda^{r+1}(M_b \oplus \Lambda^r M_b^*)$ over B_b along the isomorphism $\Lambda^{r+1}\gamma'$. Since

$$\Lambda^{r+1}(M_b \oplus \Lambda^r M_b^*) \xrightarrow{\phi} B_b \quad \text{over } B_b,$$

$\Lambda^{r+1}P'$ is isomorphic to a projective B -module L' which is given by gluing B_{1+b} over B_{1+b} and B_b over B_b , along with the isomorphism $\psi' = \phi_{1+b} \circ \Lambda^{r+1}\gamma' : B_{b(1+b)}^{\oplus r+1} \xrightarrow{\Lambda^{r+1}\gamma'} \Lambda^{r+1}(M_{b(1+b)} \oplus \Lambda^r M_{b(1+b)}^*) \xrightarrow{\phi_{1+b}} B_{b(1+b)}$. However, $\psi' = \phi_{1+b} \circ \Lambda^{r+1}\gamma' = \phi_{1+b} \circ \Lambda^{r+1}\gamma \cdot \psi^{-1} = \psi \cdot \psi^{-1} = 1$ this time. Therefore, $P' \simeq B^{\oplus r+1}$. P' maps onto M says that $\mu(M) \leq r+1$.

Remark. (1) To prove the theorem, at one point López had to use Lindel's theorem about extendibility of a projective module over a polynomial ring, which requires the ground ring to be essentially a regular affine ring. That is why he had to put the affineness condition into his theorem.

(2) There are many noetherian rings which are universally catenary and equidimensional but not necessarily affine. For example, any quotient ring of a Cohen-Macaulay ring is universally catenary [M, Theorem 17.9].

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