MULTIPLE CANONICAL DECOMPOSITIONS OF FAMILIES OF OPERATORS AND A MODEL OF QUASINORMAL FAMILIES

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Dedicated to the memory of our friend Domingo Herrero

ABSTRACT. A general method of canonical decompositions of several operator-valued functions (operators) is presented. A model of a family of doubly commuting quasinormal operators is constructed.

1. INTRODUCTION

The idea of decomposing an operator (a family of operators, an operator-valued function) into parts, which are easier to investigate than the original operator, is fundamental to the theory of operators.

The so-called canonical decomposition is one of many known kinds of decompositions. Wold [11], studying stationary stochastic processes, discovered the decomposition of an isometry into the unitary and the completely nonunitary parts, which has since been referred to as the Wold decomposition of an isometry and is the first example of a canonical decomposition. A general theory of canonical decompositions of a single operator (more generally, an operator-valued function) with respect to a single property has been developed in [7]. Following Wold's idea, Śłociński [6] studied canonical decompositions of two commuting isometries. Using in an essential manner the isometric-unitary properties, he established a criterion for "the best" analogue of the Wold decomposition to exist and found an ingenious example of two commuting isometries without such decomposition [6, Example 1].

In §2 we present a general method of constructing canonical decompositions of several operator-valued functions (operators, in particular) with respect to several properties, which, in order to make such decompositions possible, not only should behave well themselves (as in [7]) but also should behave well with respect to each other (Theorem 2.2). This multiple canonical decomposition method is based on results of [7].

Canonical decompositions are often the first step in constructing models of operators. Of particular interest here is the model of a single quasinormal

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operator discovered by Brown in [2]. This model inspired §3, in which, using a multiple canonical decomposition of a family of doubly commuting operators (Corollary 2.8) together with Ślociński's complete analysis of doubly commuting shifts [6], we establish a model of a family of doubly commuting quasinormal operators.

$B(H)$ stands for the algebra of all linear, bounded operators in a Hilbert space H . I denotes the identity operator. A *projection* is an operator $E \in B(H)$ that satisfies $E = E^2 = E^*$. If $\mathcal{S} \subset B(H)$, then $\mathcal{S}'$ denotes the commutant of $\mathcal{S}$ and $W^*(\mathcal{S})$ denotes the von Neumann algebra generated by $\mathcal{S}$. Two projections E, F in a von Neumann algebra $\mathfrak{A} \subset B(H)$ are *equivalent* with respect to $\mathfrak{A}$ if there is a partial isometry $V \in \mathfrak{A}$ such that $V^*V = E$, $VV^* = F$. A family $(T_j)_{j \in J} \subset B(H)$ is *unitarily equivalent* to a family $(S_j)_{j \in J} \subset B(K)$, where K is a Hilbert space, if there is a unitary mapping $\phi: K \rightarrow H$ such that $\phi S_j = T_j \phi$, $j \in J$. $\mathbb{N}$ stands for the set of all nonnegative integers.

2. MULTIPLE CANONICAL DECOMPOSITIONS

First the terminology and necessary results will be recalled from [7].

Let Ω be a set, let $\varphi: \Omega \rightarrow B(H)$ be a function, and let (W) be a property concerning φ . A projection $E \in B(H)$ is called (W) -*reducing* for φ if $E \in \varphi(\Omega)'$ and the function $u \rightarrow \varphi(u)|_{EH}$ has the property (W) . Denote by $\mathcal{P}$ the family of all (W) -reducing projections for φ . A property (W) is called *hereditary* if, for each $E \in \mathcal{P}$, for each projection $F \in \varphi(\Omega)'$, $F \leq E$ implies $F \in \mathcal{P}$. Let $Q \in \varphi(\Omega)'$ be a projection. φ is called *c.n. (W)* (*completely non-(W)*) on QH if the only subprojection of Q that (W) -reduces φ is zero.

(2.1) **Theorem** (cf. [7, Proposition 1, Theorem 5]). *If (W) is hereditary and $\mathcal{P}$ is strongly closed and satisfies the conditions*

- (a) *if $E, F \in \mathcal{P}$, $EF = 0$, then $E + F \in \mathcal{P}$, and*
- (b) *if $E \in \mathcal{P}$ and $F \in \varphi(\Omega)'$ is a projection equivalent to E with respect to $W^*(\varphi(\Omega))'$, then $F \in \mathcal{P}$,*

then $P = \sup \mathcal{P}$ exists, P is the largest (W) -reducing projection for φ , and P belongs to the center of $W^(\varphi(\Omega))$. Moreover, φ is c.n. (W) on $(I - P)H$, and the decomposition $\varphi(u) = \varphi(u)|_{PH} \oplus \varphi(u)|_{(I-P)H}$ is unique.*

The above decomposition is called the (W) -canonical decomposition of φ .

In what follows canonical decompositions will be constructed for an arbitrary number of functions and an arbitrary number of properties. The number of functions does not need to equal the number of properties. To make the exposition clear, attention will be restricted to two functions and two properties.

Let Ω_j be a set, let $\varphi_j: \Omega_j \rightarrow B(H)$ be a function, and let (W_j) be a property concerning φ_j , $j = 1, 2$. One shall say that φ_1, φ_2 have the (W_1, W_2) -canonical decomposition if there exist projections $E_1, E_2, E_3, E_4 \in B(H)$ such that

- (C1) $E_k \in (\varphi_1(\Omega_1) \cup \varphi_2(\Omega_2))'$, $k = 1, \dots, 4$;
- (C2) $E_i E_k = 0$ if $i \neq k$;
- (C3) $I = E_1 + E_2 + E_3 + E_4$;
- (C4) E_1 is a (W_j) -reducing projection for φ_j , $j = 1, 2$;
- (C5) E_2 is a (W_1) -reducing projection for φ_1 and φ_2 is c.n. (W_2) on $E_2 H$;

- (C6) φ_1 is c.n. (W_1) on E_3H and E_3 is a (W_2) -reducing projection for φ_2 ;
 (C7) φ_j is c.n. (W_j) on E_4H , $j = 1, 2$.

This notation will be preserved. The following theorem is the main result of this section.

(2.2) **Theorem.** *Let Ω_j be a set, let $\varphi_j : \Omega_j \rightarrow B(H)$ be a function, and let (W_j) be a hereditary property concerning φ_j , $j = 1, 2$. Suppose that there exists the largest (W_j) -reducing projection P_j for φ_j and that P_j is in the center of $W^*(\varphi_j(\Omega_j))$, $j = 1, 2$. Then φ_1, φ_2 have the (W_1, W_2) -canonical decomposition if and only if $P_j \in (\varphi_1(\Omega_1) \cup \varphi_2(\Omega_2))'$, $j = 1, 2$. In such a case, $P_1 = E_1 + E_2$, $P_2 = E_1 + E_3$, and $E_1 = P_1P_2$, $E_2 = P_1(I - P_2)$, $E_3 = (I - P_1)P_2$, $E_4 = (I - P_1)(I - P_2)$. The decomposition is unique.*

Proof. Uniqueness follows from the uniqueness of P_1, P_2 for φ_1, φ_2 , respectively, by Theorem 2.1 and from the last four equalities. It should be mentioned that hereditariness is the main property responsible for uniqueness (cf. [7, (A)]). If $P_1, P_2 \in (\varphi_1(\Omega_1) \cup \varphi_2(\Omega_2))'$, then $P_1P_2 = P_2P_1$, because P_j is in the center of $W^*(\varphi_j(\Omega_j))$, $j = 1, 2$. The last four equalities in the statement of the theorem now define projections that satisfy (C1)–(C7).

Conversely, suppose that φ_1, φ_2 have the (W_1, W_2) -canonical decomposition. Note that $E_kP_j = P_jE_k$, $j = 1, 2$, $k = 1, \dots, 4$. For, $P_j \in W^*(\varphi_j(\Omega_j))$ and $E_k \in W^*(\varphi_j(\Omega_j))'$ by (C1). Since E_1, E_2 are (W_1) -reducing projections for φ_1 and P_1 is the largest one with this property, $E_1 + E_2 \leq P_1$. Now, $P_1E_3 \leq P_1$ is a (W_1) -reducing projection for φ_1 , because (W_1) is hereditary. On the other hand, $P_1E_3 \leq E_3$ and φ_1 is c.n. (W_1) on E_3H . Hence $P_1E_3 = 0$. In a similar manner one proves that $P_1E_4 = 0$. Thus, by (C3), $P_1 \leq E_1 + E_2$. An analogous argument justifies that $P_2 = E_1 + E_3$. Hence $P_j \in (\varphi_1(\Omega_1) \cup \varphi_2(\Omega_2))'$ by (C1). $\square$

(2.3) **Corollary.** *Let Ω_j , φ_j , (W_j) , and P_j be as in Theorem (2.2). Suppose $P_j \in W^*(\varphi_j(\Omega_j))$, $j = 1, 2$. If $\varphi_1(\Omega_1) \subset W^*(\varphi_2(\Omega_2))'$, then φ_1, φ_2 have the (W_1, W_2) -canonical decomposition.*

Proof. It is clear that $W^*(\varphi_1(\Omega_1)) \subset W^*(\varphi_2(\Omega_2))'$. Also, by the Double Commutant Theorem $W^*(\varphi_2(\Omega_2)) \subset W^*(\varphi_1(\Omega_1))'$. Since $P_j \in W^*(\varphi_j(\Omega_j))$, one concludes that $P_j \in (W^*(\varphi_1(\Omega_1)) \cup W^*(\varphi_2(\Omega_2)))'$, $j = 1, 2$. Use Theorem (2.2). $\square$

For one function φ ($= \varphi_1 = \varphi_2$) and several properties one immediately has

(2.4) **Corollary.** *Let $\varphi : \Omega \rightarrow B(H)$ be a function, and let $(W_1), (W_2)$ be hereditary properties concerning φ . If there exists the largest (W_j) -reducing projection P_j for φ and P_j is in the center of $W^*(\varphi(\Omega))$, $j = 1, 2$, then φ has the (W_1, W_2) -canonical decomposition.*

Proof. Obvious by Theorem (2.2), because $P_1, P_2 \in \varphi(\Omega)'$. $\square$

In an even more particular case of $\varphi_1 = \varphi_2$ and $(W_1) = (W_2)$, the (W_1, W_2) -canonical decomposition of φ_1, φ_2 reduces to the (W_1) -canonical decomposition of φ_1 . It is known that the (W) -canonical decomposition of φ sometimes does not exist (cf., e.g., [9] for $(W) = (\text{symmetric commutant})$, [8] for $(W) = (\text{antisymmetric})$). Thus the (W_1, W_2) -canonical decomposition of φ_1, φ_2 may not exist either. An example of two functions (operators) for which

the (W, W) -canonical decomposition does not exist, where $(W) = (\text{unitary})$, even though the individual (W) -canonical decomposition (in this case, plainly, the Wold decomposition of an isometry) exists for each function (operator), can be found in [6, Example 1]. In this example the assumption $P_j \in (\varphi_1(\Omega_1) \cup \varphi_2(\Omega_2))'$, $j = 1, 2$, of Theorem (2.2) fails to be satisfied.

To explain how to use the above results for operators rather than operator-valued functions, consider the following particular situation. Let $T_1, T_2 \in B(H)$. Let Ω be an arbitrary set, and let $\varphi_j : \Omega \rightarrow B(H)$ be the constant function defined by $\varphi_j(u) = T_j$, $u \in \Omega$, $j = 1, 2$. The operators T_1, T_2 will be said to have the (W_1, W_2) -canonical decomposition if the functions φ_1, φ_2 have the (W_1, W_2) -canonical decomposition. In this simpler case, often sufficient and more convenient for applications to operators (e.g., in the next section), Theorem (2.2) reads as follows:

(2.5) **Corollary.** *Let $T_j \in B(H)$, and let (W_j) be a property concerning T_j , $j = 1, 2$. Suppose there is the largest (W_j) -reducing projection P_j for T_j and $P_j \in W^*(T_j)$, $j = 1, 2$. Then T_1, T_2 have the (W_1, W_2) -canonical decomposition if and only if $P_j \in \{T_1, T_2\}'$, $j = 1, 2$.*

Operators $T_1, T_2 \in B(H)$ doubly commute if $T_1 \in W^*(T_2)'$ or, equivalently, $W^*(T_1) \subset W^*(T_2)'$. Corollary (2.3) interpreted for operators concerns exactly this situation and reads

(2.6) **Corollary.** *Let T_j , (W_j) , and P_j be as in Corollary (2.5). Suppose $P_j \in W^*(T_j)$, $j = 1, 2$. If T_1, T_2 doubly commute, then T_1, T_2 have the (W_1, W_2) -canonical decomposition.*

The existence of the (W_1, W_2) -canonical decomposition does not imply, however, that the operators doubly commute. Here is an example.

(2.7) **Example.** Take a nonunitary isometry $V \in B(H)$, $T_1 = T_2 = V$, and $(W_1) = (W_2) = (\text{unitary})$. Then the (W_1, W_2) -canonical decomposition of T_1, T_2 exists, and it is simply the Wold decomposition of V . All assumptions of Corollary (2.5) are satisfied because the largest (unitarily)-reducing projection P for V belongs to $W^*(V)$ by Theorem (2.1) and, clearly, $P_1 = P_2 = P \in \{T_1, T_2\}'$. Since V is not unitary, T_1, T_2 do not doubly commute.

The interpretation of Corollary (2.4) for a single operator and several properties is left to the reader.

In general, to check whether there is the largest (W) -reducing projection P for $T \in B(H)$ and $P \in W^*(T)$, one uses Theorem (2.1). For example, if p is a polynomial in two noncommuting variables and the property (W) is defined by the condition $p(S, S^*) = 0$ (or $p(S, S^*) \geq 0$) for Hilbert space operators S , then the assumptions of Theorem (2.1) are satisfied, which can be deduced from results of [7] (cf. [4]). In particular, let

$$p(S, S^*) = S^*S - SS^*.$$

Then Corollary (2.6) implies the following result needed in the next section:

(2.8) **Corollary.** *If $T_1, T_2 \in B(H)$ doubly commute, then they have the (normal, normal)-canonical decomposition.*

3. A MODEL OF A FAMILY OF DOUBLY COMMUTING QUASINORMAL OPERATORS

In this section we are going to construct a model of two doubly commuting quasinormal operators. For the sake of clarity we restrict our attention to two operators only. Our model, however, can be written up for any number of doubly commuting quasinormal operators.

Recall that an operator $T \in B(H)$ is *quasinormal* if T commutes with T^*T . In the literature, for example [5, 10], jointly quasinormal families of operators are also considered. A family Q of operators is called *jointly quasinormal* if C commutes with D and D^*D for all C and D in Q . A family of quasinormal operators that doubly commute is a jointly quasinormal family (cf. Lemma (3.3)). The opposite, however, fails, which can be seen considering two commuting isometries that do not doubly commute.

Unfortunately, we are not able to present a model for every quasinormal family, because our model relies on the canonical decomposition of families of operators developed in §2. Namely, jointly quasinormal families need not have (normal, normal, ...)-canonical decomposition. To see this, consider Ślocinski's Example 1 in [6]. The two isometries in this example commute; thus, they are a jointly quasinormal family, but they do not have the (normal, normal)-canonical decomposition, (cf. remarks after Corollary (2.4)).

Let us consider quasinormal operators $T_1, T_2 \in B(H)$ and assume that they doubly commute. According to Corollary (2.8) they have the (normal, normal)-canonical decomposition. Therefore, in order to construct a model of these operators it is enough to consider two cases:

Case 1. T_1 is a normal operator, and T_2 is a c.n. normal operator.

Case 2. Both T_1 and T_2 are c.n. normal operators. Recall that an isometry $V \in B(H)$ is a *shift* (a c.n. unitary isometry) if $V^m M \perp V^n M$ for $m \neq n$, $m, n \in \mathbb{N}$, and $H = \bigoplus_{n \in \mathbb{N}} V^n M$, where $M = (VH)^\perp$. Throughout the remainder of this paper l_+^2 is the Hilbert space of all square summable complex sequences with the canonical base $\{e_0, e_1, e_2, \dots\}$, and U_+ is the unilateral shift of multiplicity one in l_+^2 .

Case 1.

(3.1) **Proposition.** *Let $T_1 \in B(H)$ be a c.n. normal, quasinormal operator, and let $T_2 \in B(H)$ be a normal operator commuting with T_1 . Then there exist a Hilbert space M , a positive operator $Q \in B(M)$ with $\ker Q = \{0\}$, and a normal operator $N \in B(M)$ commuting with Q such that the pair $T_1, T_2 \in B(H)$ is unitarily equivalent to the pair $U_+ \otimes Q, I \otimes N \in B(l_+^2 \otimes M)$.*

Proof. Let $T_1 = VP$ be the polar decomposition of T_1 . Since T_1 is quasinormal and c.n. normal, $\ker T_1 = \{0\}$. Thus $\ker P = \{0\}$ and V is an isometry. Moreover, V is a shift (cf. [2] or [3, Chapter II, proof of Theorem 3.2]). Let $M = (VH)^\perp$. Since V is a shift, the mapping $\phi : l_+^2 \otimes M \rightarrow H$ defined by $\phi(\sum_{n \in \mathbb{N}} e_n \otimes x_n) = \sum_{n \in \mathbb{N}} V^n x_n$, $x_n \in M$, $n \in \mathbb{N}$, is unitary. It is known that $\phi(U_+ \otimes I) = V\phi$. Since T_1 is quasinormal, P commutes with V , and thus M reduces P . Let $Q = P|_M$. One checks easily that $\phi(I \otimes Q) = P\phi$. Until now we have merely given a short proof of the main result of [2, Theorem 1]. For a proof that does not use the tensor product language the reader may consult [3, Chapter II, Theorem 3.2].

Now let us turn to T_2 which is a normal operator commuting with T_1 . By Fuglede's theorem $T_2 \in W^*(T_1)'$. It is known that

(3.2) *Both factors in the polar decomposition of an operator belong to the von Neumann algebra generated by that operator,*

which, for the partial isometry factor, is a consequence of the Double Commutant Theorem (cf., e.g., [1, Corollary, p. 7]). Hence T_2 commutes with V and V^* , and thus M reduces T_2 . Let $N = T_2|_M$. Since T_2 commutes also with P , N commutes with Q . Also, $\ker Q = \{0\}$, because $\ker P = \{0\}$. Finally, if $x_n \in M$, $n \in \mathbb{N}$, then

$$\begin{aligned} \phi(I \otimes N) \left(\sum e_n \otimes x_n \right) &= \phi \left(\sum e_n \otimes N x_n \right) = \sum V^n N x_n = T_2 \sum V^n x_n \\ &= T_2 \phi \left(\sum e_n \otimes x_n \right), \end{aligned}$$

which proves that $\phi(I \otimes N) = T_2 \phi$. $\square$

The next lemma, which will be needed in Case 2, is of interest on its own, because it clarifies the distinction between doubly commuting quasinormal operators and a jointly quasinormal family.

(3.3) **Lemma.** *Let $T_j = V_j P_j$ be a polar decomposition, unique under the assumption $\ker V_j = \ker P_j$, of an operator $T_j \in B(H)$, $j = 1, 2$. The following conditions are equivalent.*

- (a) T_1, T_2 are quasinormal, and they doubly commute;
- (b) T_1, T_2 are jointly quasinormal, and $V_1 V_2^* = V_2^* V_1$;
- (c) V_1, V_2, P_1, P_2 commute each other, and $V_1 V_2^* = V_2^* V_1$.

Proof. (b) $\Leftrightarrow$ (c) in Lemma 3.9(b) of [10] (cf. also [5, proof of Theorem 4.5]). (c) $\Rightarrow$ (a) is clear. Now we prove (a) $\Rightarrow$ (c). $V_j P_j = P_j V_j$, because T_j is quasinormal, $j = 1, 2$. Since T_1, T_2 doubly commute, $W^*(T_1) \subset W^*(T_2)'$. By (3.2) we conclude that V_1, V_2, P_1, P_2 commute each other.

Moreover, $V_1 V_2^* P_1 P_2 = V_1 P_1 P_2 V_2^* = T_1 T_2^* = T_2^* T_1 = P_2 V_2^* V_1 P_1 = V_2^* V_1 P_1 P_2$, which proves that $V_1 V_2^* x = V_2^* V_1 x$ if $x \in (P_1 P_2 H)^\perp =$ the closure of $P_1 P_2 H = (\ker P_1 P_2)^\perp$.

Now we shall prove that $V_1 V_2^* x = V_2^* V_1 x = 0$ for $x \in \ker P_1 P_2$. Since T_j is quasinormal, $\ker T_j = \ker P_j$ reduces T_j , $j = 1, 2$. Since T_1, T_2 doubly commute, $\ker P_j$ reduces T_i , $i, j = 1, 2$. By (3.2) again, $\ker P_j$ reduces V_i , $i, j = 1, 2$.

Let $x \in \ker P_1 = \ker V_1$. Then $V_1 x = 0$ and $V_2^* x \in \ker P_1 = \ker V_1$. Thus $V_1 V_2^* x = V_2^* V_1 x = 0$, i.e., $V_1 V_2^* = V_2^* V_1 = 0$ on $\ker P_1$.

One proves similarly that $V_1^* V_2 = V_2 V_1^* = 0$ on $\ker P_2$, which is equivalent to $V_2^* V_1 = V_1 V_2^* = 0$ on $\ker P_2$, because $\ker P_2$ reduces V_1, V_2 . Since P_1, P_2 are commuting positive operators, it follows from the spectral theorem that $\ker P_1 P_2 = (\ker P_1 + \ker P_2)^\perp$, which completes the proof. $\square$

Case 2. We begin our discussion of Case 2 with the following

(3.4) **Proposition.** *Let $V_1, V_2 \in B(H)$ be shifts that doubly commute. Let $M = (V_1 H)^\perp \cap (V_2 H)^\perp$. Then the pair $V_1, V_2 \in B(H)$ is unitarily equivalent to the pair $U_+ \otimes I \otimes I, I \otimes U_+ \otimes I \in B(l_+^2 \otimes l_+^2 \otimes M)$.*

Proof. Recall that the set $\{e_m \otimes e_n : m, n \in \mathbb{N}\}$ is an orthonormal base of $l_+^2 \otimes l_+^2$. It follows from Theorem 1 of [6] that $V_1^k V_2^l M \perp V_1^m V_2^n M$ for $(k, l) \neq (m, n)$, $k, l, m, n \in \mathbb{N}$, and that $H = \bigoplus_{m, n \in \mathbb{N}} V_1^m V_2^n M$. Hence the mapping $\psi : l_+^2 \otimes l_+^2 \otimes M \rightarrow H$ defined by

$$\psi \left(\sum_{m, n \in \mathbb{N}} e_m \otimes e_n \otimes x_{mn} \right) = \sum_{m, n \in \mathbb{N}} V_1^m V_2^n x_{mn}, \quad x_{mn} \in M,$$

is unitary. Take $x_{mn} \in M$, $m, n \in \mathbb{N}$, and define $y_{mn} = x_{m-1, n}$ if $m > 0$, $y_{0n} = 0$, $n \in \mathbb{N}$. Then

$$\begin{aligned} \psi(U_+ \otimes I \otimes I) \left(\sum e_m \otimes e_n \otimes x_{mn} \right) &= \psi \left(\sum e_{m+1} \otimes e_n \otimes x_{mn} \right) \\ &= \psi \left(\sum e_{m+1} \otimes e_n \otimes y_{m+1, n} \right) = \psi \left(\sum e_m \otimes e_n \otimes y_{mn} \right) \\ &= \sum V_1^m V_2^n y_{mn} = \sum V_1^{m+1} V_2^n y_{m+1, n} = \sum V_1^{m+1} V_2^n x_{mn} \\ &= V_1 \sum V_1^m V_2^n x_{mn} = V_1 \psi \left(\sum e_m \otimes e_n \otimes x_{mn} \right), \end{aligned}$$

which proves that $\psi(U_+ \otimes I \otimes I) = V_1 \psi$. In a similar manner one proves that $\psi(I \otimes U_+ \otimes I) = V_2 \psi$, “shifting” the second index in x_{mn} and, at the end, using again the assumption that V_1, V_2 (doubly) commute. $\square$

(3.5) Proposition. Let $T_1, T_2 \in B(H)$ be c.n. normal, quasinormal operators that doubly commute. Then there exist a Hilbert space M and positive, commuting operators $Q_1, Q_2 \in B(M)$ with $\ker Q_j = \{0\}$, $j = 1, 2$, such that the pair $T_1, T_2 \in B(H)$ is unitarily equivalent to the pair $U_+ \otimes I \otimes Q_1, I \otimes U_+ \otimes Q_2 \in B(l_+^2 \otimes l_+^2 \otimes M)$.

Proof. Let $T_j = V_j P_j$ be the polar decomposition of T_j . Then $\ker P_j = \{0\}$ and V_j is a shift, $j = 1, 2$ (cf. the beginning of the proof of Proposition (3.1)). Since T_1, T_2 doubly commute, the operators P_1, P_2, V_1, V_2 commute each other (cf. Lemma (3.3)). Hence $M = (V_1 H)^\perp \cap (V_2 H)^\perp$ reduces P_j . Let $Q_j = P_j|_M$. Clearly, $\ker Q_j = \{0\}$, $j = 1, 2$, and Q_1 and Q_2 commute. Now apply Proposition (3.4) to V_1, V_2 , which doubly commute, by Lemma (3.3). Let $\psi : l_+^2 \otimes l_+^2 \otimes M \rightarrow H$ be the unitary mapping defined in the proof of Proposition (3.4). Take $x_{mn} \in M$, $m, n \in \mathbb{N}$. Then

$$\begin{aligned} \psi(I \otimes I \otimes Q_j) \left(\sum e_m \otimes e_n \otimes x_{mn} \right) &= \psi \left(\sum e_m \otimes e_n \otimes Q_j x_{mn} \right) \\ &= \sum V_1^m V_2^n Q_j x_{mn} = P_j \sum V_1^m V_2^n x_{mn} = P_j \psi \left(\sum e_m \otimes e_n \otimes x_{mn} \right), \end{aligned}$$

which proves that $\psi(I \otimes I \otimes Q_j) = P_j \psi$, $j = 1, 2$.

This, together with Proposition (3.4), finishes the proof. $\square$

The final result summarizes all the above results and gives a complete model of two doubly commuting quasinormal operators.

(3.6) Theorem. Let T_1 and T_2 be doubly commuting quasinormal operators. Then the pair T_1, T_2 is unitarily equivalent to the pair

$$\begin{aligned} M_1 \oplus (I \otimes N_1) \oplus (U_+ \otimes P_1) \oplus (U_+ \otimes I \otimes Q_1), \\ M_2 \oplus (U_+ \otimes P_2) \oplus (I \otimes N_2) \oplus (I \otimes U_+ \otimes Q_2), \end{aligned}$$

where

- (a) M_1 , M_2 , N_1 , and N_2 are normal operators,
- (b) P_1 , P_2 , Q_1 , and Q_2 are positive operators with zero kernels,
- (c) M_1 and M_2 ; P_1 and N_2 ; N_1 and P_2 ; and Q_1 and Q_2 commute.

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