ON THE LEFT IDEAL IN THE UNIVERSAL ENVELOPING ALGEBRA OF A LIE GROUP GENERATED BY A COMPLEX LIE SUBALGEBRA

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ABSTRACT. Let G_0 be a connected Lie group with Lie algebra $\mathfrak{g}_0$ and let $\mathfrak{h}$ be a Lie subalgebra of the complexification $\mathfrak{g}$ of $\mathfrak{g}_0$. Let $C^\infty(G_0)^{\mathfrak{h}}$ be the annihilator of $\mathfrak{h}$ in $C^\infty(G_0)$ and let $\mathcal{A} = \mathcal{A}(C^\infty(G_0)^{\mathfrak{h}})$ be the annihilator of $C^\infty(G_0)^{\mathfrak{h}}$ in the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$. If $\mathfrak{h}$ is the complexification of the Lie algebra $\mathfrak{h}_0$ of a Lie subgroup H_0 of G_0 then $\mathcal{A} = \mathcal{U}(\mathfrak{g})\mathfrak{h}$ whenever H_0 is closed, is a known result, and the point of this paper is to prove the converse assertion. The paper has two distinct parts, one for C^∞ , the other for holomorphic functions. In the first part the Lie algebra $\bar{\mathfrak{h}}_0$ of the closure of H_0 is characterized as the annihilator in $\mathfrak{g}_0$ of $C^\infty(G_0)^{\mathfrak{h}}$, and it is proved that $\mathfrak{h}_0$ is an ideal in $\bar{\mathfrak{h}}_0$ and that $\bar{\mathfrak{h}}_0 = \mathfrak{h}_0 \oplus \mathfrak{v}$ where $\mathfrak{v}$ is an abelian subalgebra of $\bar{\mathfrak{h}}_0$. In the second part we consider a complexification G of G_0 and assume that $\mathfrak{h}$ is the Lie algebra of a closed connected subgroup H of G . Then we establish that $\mathcal{A}(\mathcal{O}(G)^{\mathfrak{h}}) = \mathcal{U}(\mathfrak{g})\mathfrak{h}$ if and only if G/H has many holomorphic functions. This is the case if G/H is a quasi-affine variety. From this we get that if H is a unipotent subgroup of G or if G and H are reductive groups then $\mathcal{A}(C^\infty(G_0)^{\mathfrak{h}}) = \mathcal{U}(\mathfrak{g})\mathfrak{h}$.

1. THE SMOOTH SETTING

Let G_0 be a (connected) Lie group with Lie algebra $\mathfrak{g}_0$. Let $\mathfrak{h}$ be a Lie subalgebra of the complexification $\mathfrak{g}$ of $\mathfrak{g}_0$. Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ viewed as the algebra of all left invariant differential operators on G_0 . Hence the space $C^\infty(G_0)$ of all complex-valued C^∞ -functions on G_0 is a left $\mathcal{U}(\mathfrak{g})$ -module. If s is any subset of $\mathfrak{g}$ we set

$$C^\infty(G_0)^s = \{f \in C^\infty(G_0) : Xf = 0 \text{ for all } x \in s\}.$$

We shall be concerned with the annihilator $\mathcal{A} = \mathcal{A}(C^\infty(G_0)^{\mathfrak{h}})$ in $\mathcal{U}(\mathfrak{g})$ of the annihilator $C^\infty(G_0)^{\mathfrak{h}}$ of $\mathfrak{h}$ in $C^\infty(G_0)$. Thus

$$\mathcal{A} = \{D \in \mathcal{U}(\mathfrak{g}) : Df = 0 \text{ for all } f \in C^\infty(G_0)^{\mathfrak{h}}\}.$$

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Clearly $\mathcal{A}$ is a left ideal of $\mathcal{U}(g)$ containing the left ideal $\mathcal{U}(g)h$ generated by h . We are interested in discussing when $\mathcal{A}$ is equal to $\mathcal{U}(g)h$.

Suppose h is the complexification of $h_0 = h \cap g_0$, h_0 being the Lie algebra of a closed connected subgroup H_0 of G_0 . Then it is easy to see that in this case $\mathcal{A} = \mathcal{U}(g)h$. Let us go over the proof of this fact (cf. [5, Lemma 2.5, Chapter X]), because we shall make use of a similar argument.

If V is a finite-dimensional complex vector space, $S(V)$ will denote the symmetric algebra over V and λ will denote the symmetrization mapping of $S(g)$ onto $\mathcal{U}(g)$. Let $m_0 \subset g_0$ be a subspace such that $g_0 = m_0 \oplus h_0$ and let $m \subset g$ be its complexification. Then it is well known (cf. [3, Proposition 2.4.14, p. 82]) that λ induces a linear isomorphism between $S(m) \otimes S(h)$ and $\mathcal{U}(g)$. Therefore we have

$$(1) \quad \begin{aligned} \mathcal{U}(g) &= \lambda(S(m)) \cdot \lambda(S(h)) \\ &= \lambda(S(m)) \cdot (\mathcal{U}(h)h \oplus \mathbb{C} \cdot 1) = \mathcal{U}(g)h \oplus \lambda(S(m)). \end{aligned}$$

Let $X_1, \dots, X_p$ be a basis of g_0 such that $X_i \in m_0$ for $i = 1, \dots, n$. If π denotes the natural mapping of G_0 onto G_0/H_0 , for a suitable open neighbourhood U of 0 in m_0 , the mapping

$$\pi(\exp(x_1 X_1 + \dots + x_n X_n)) \mapsto (x_1, \dots, x_n)$$

is a coordinate system on the open neighbourhood $\pi(\exp U)$ of the origin o in G_0/H_0 .

If $P \in S(g)$ is identified with a polynomial in the basis elements $X_1, \dots, X_p$ and $\partial_1, \dots, \partial_p$ denote the partial derivatives with respect to the variables $(x_1, \dots, x_p)$, then (cf. [5, Chapter X, §2, 16])

$$(2) \quad [\lambda(P)f](g) = [P(\partial_1, \dots, \partial_p)f(g \exp(x_1 X_1 + \dots + x_p X_p))](0)$$

for each $g \in G_0$, $f \in C^\infty(G_0)$.

In order to prove that $\mathcal{A}(C^\infty(G_0)^h) = \mathcal{U}(g)h$ let $P \in S(m)$, $P \neq 0$. There exists $f^* \in C^\infty(U)$ such that

$$[P(\partial_1, \dots, \partial_n)f^*(x_1 X_1 + \dots + x_n X_n)](0) \neq 0.$$

Using a partition of unity argument, it is clear that there exists a function $f \in C^\infty(G_0/H_0)$ such that $f(\pi(\exp(x_1 X_1 + \dots + x_n X_n))) = f^*(x_1 X_1 + \dots + x_n X_n)$ for all sufficiently small x_i . From (2) follows

$$[\lambda(P)(f \circ \pi)](e) = [P(\partial_1, \dots, \partial_n)f^*(x_1 X_1 + \dots + x_n X_n)](0) \neq 0,$$

but since $(f \circ \pi) \in C^\infty(G_0)^h$ it is clear from (1) that $\mathcal{A} = \mathcal{U}(g)h$.

Thus we have

Theorem 1. *Suppose h is the complexification of the Lie algebra of a closed connected Lie subgroup H_0 of G_0 . Then*

$$\mathcal{A}(C^\infty(G_0)^h) = \mathcal{U}(g)h.$$

Corollary 2. *Suppose h is the complexification of the Lie algebra of a connected Lie subgroup H_0 of G_0 . Let $\bar{h}$ be the complexification of the Lie algebra of the closure $\bar{H}_0$ of H_0 . Then*

$$\mathcal{A}(C^\infty(G_0)^h) = \mathcal{U}(g)\bar{h}.$$

Proof. If K_0 is a subgroup of G_0 we denote by $C^\infty(G_0)^{K_0}$ the space of all C^∞ -functions on G_0 which are right invariant by all elements in K_0 . It is easy to see that

$$C^\infty(G_0)^h = C^\infty(G_0)^{H_0} = C^\infty(G_0)^{\bar{H}_0} = C^\infty(G_0)^{\bar{h}}.$$

Thus

$$\mathcal{A}(C^\infty(G_0)^h) = \mathcal{A}(C^\infty(G_0)^{\bar{h}}) = \mathcal{U}(g)\bar{h}.$$

This completes the proof of the corollary.

If $\mathcal{F}$ is any subset of $C^\infty(G_0)$ we define

$$a_0(\mathcal{F}) = \{X \in g_0 : Xf = 0 \text{ for all } f \in \mathcal{F}\}.$$

It is easy to verify that $a_0(\mathcal{F})$ is a Lie subalgebra of g_0 .

Corollary 3. *A Lie subalgebra h_0 of g_0 is the Lie algebra of a closed Lie subgroup of G_0 if and only if*

$$a_0(C^\infty(G_0)^{h_0}) = h_0.$$

Proof. Let h be the complexification of h_0 . Then

$$a_0(C^\infty(G_0)^{h_0}) = \mathcal{A}(C^\infty(G_0)^h) \cap g_0.$$

Let H_0 be the connected Lie subgroup of G_0 with Lie algebra h_0 and let $\bar{h}_0$ be the Lie algebra of its closure $\bar{H}_0$. Then, by Corollary 2, we have

$$(3) \quad a_0(C^\infty(G_0)^{h_0}) = \mathcal{U}(g)\bar{h} \cap g_0 = \bar{h}_0.$$

Thus H_0 is closed if and only if $a_0(C^\infty(G_0)^{h_0}) = h_0$. Since the connected component of a closed subgroup is closed, the corollary is proved.

Definition. If h_0 is a Lie subalgebra of g_0 we set $\bar{h}_0 = a_0(C^\infty(G_0)^{h_0})$.

Then as it was pointed out above $\bar{h}_0$ is the Lie algebra of the closure $\bar{H}_0$ of the connected Lie subgroup H_0 corresponding to h_0 .

Corollary 4. *The family of all subalgebras $a_0(\mathcal{F})$, where $\mathcal{F}$ is any subset of $C^\infty(G_0)$ is precisely the family of all subalgebras of g_0 which correspond to the closed subgroups of G_0 .*

Proof. Given $\mathcal{F} \subset C^\infty(G_0)$ let H_0 be the connected Lie subgroup of G_0 with Lie algebra $h_0 = a_0(\mathcal{F})$. Since $\mathcal{F} \subset C^\infty(G_0)^{h_0}$ we have

$$h_0 \supseteq a_0(C^\infty(G_0)^{h_0}) \supseteq h_0.$$

Thus from Corollary 3 it follows that H_0 is closed.

Conversely if h_0 is the Lie algebra of a closed subgroup of G_0 then, by Corollary 3, $h_0 = a_0(C^\infty(G_0)^{h_0})$ which completes the proof of our statement.

In order to emphasize the characterization of the Lie algebra of the closure of a connected Lie subgroup H_0 of G_0 as the annihilator in g_0 of $C^\infty(G_0)^{h_0}$, we will consider some examples.

Example 1. Let $G_0 = \mathbb{R}^2/\mathbb{Z}^2$ be a two-dimensional torus. We identify the Lie algebra of G_0 with $\mathbb{R}^2$ and we denote with π the natural mapping of $\mathbb{R}^2$ onto G_0 . Given $(a, b) \in \mathbb{R}^2$, $(a, b) \neq 0$, let h_0 be the one-dimensional subalgebra of $\mathbb{R}^2$ spanned by (a, b) .

If a/b or b/a are rational numbers we shall prove, using the criterion of Corollary 3, that h_0 is the Lie algebra of a closed subgroup of G_0 . In fact, we

may assume that $a, b \in \mathbb{Z}$. There exists $f \in C^\infty(G_0)$ such that $(f \circ \pi)(x, y) = e^{2\pi i(bx - ay)}$ for all $(x, y) \in \mathbb{R}^2$. If $X = (c, d) \in \mathbb{R}^2$ then

$$[Xf](\pi(x, y)) = \left\{ \frac{d}{dt} e^{2\pi i(b(x+ct) - a(y+dt))} \right\}_{t=0} = 2\pi i(bc - ad)f(\pi(x, y)).$$

Thus $Xf = 0$ if and only if $X \in h_0$, in other words $a_0(\{f\}) = h_0$.

On the other hand if a/b is irrational we shall prove that $C^\infty(G_0)^{h_0} = \mathbb{C}$ which implies that the connected Lie subgroup of G_0 corresponding to h_0 is dense (cf. (3)). Let $X = (a, b)$ and let $f \in C^\infty(G_0)$ such that $Xf = 0$. Then $F = f \circ \pi$ satisfies $a \frac{\partial F}{\partial x} + b \frac{\partial F}{\partial y} = 0$. If we introduce the new linear coordinates (u, v) on $\mathbb{R}^2$, defined by $x = bu + av$, $y = -au + bv$, then $\frac{\partial F}{\partial v} = 0$, hence $F(u, v) = \varphi(u)$, $\varphi \in C^\infty(\mathbb{R})$. Therefore $F(x, y) = \varphi((bx - ay)/(a^2 + b^2))$ for all $(x, y) \in \mathbb{R}^2$. But φ is constant on the subgroup $\{(bm + an)/(a^2 + b^2) : m, n \in \mathbb{Z}\}$ of $\mathbb{R}$. This subgroup is either closed (and isomorphic to $\mathbb{Z}$) or it is dense in $\mathbb{R}$. In the first case the ratio of two of its nonzero elements would be rational, which is impossible since a/b is irrational. Therefore φ is constant and a fortiori f is constant.

Example 2 (cf. [2, p. 127]). Let G_0 be a connected, simply connected Lie group with Lie algebra g_0 . If h_0 is an ideal in g_0 , then the connected Lie subgroup H_0 of G_0 with Lie algebra h_0 is closed in G_0 .

In fact let K_0 be a connected Lie group with Lie algebra g_0/h_0 . Let $\sigma: G_0 \rightarrow K_0$ be the analytic homomorphism whose differential $d\sigma: g_0 \rightarrow g_0/h_0$ is the natural mapping. Let $\mathcal{F} = \{f \circ \sigma : f \in C^\infty(K_0)\}$. Then

$$\begin{aligned} X \in a_0(\mathcal{F}) \Leftrightarrow 0 &= X(f \circ \sigma) = d\sigma(X)f \quad \text{for all } f \in C^\infty(K_0) \\ &\Leftrightarrow d\sigma(X) = 0 \Leftrightarrow X \in h_0. \end{aligned}$$

Thus from Corollary 4 it follows that H_0 is closed in G_0 .

Example 3. Let G_0 be a connected Lie group with Lie algebra g_0 and let $\tau: G_0 \rightarrow \text{GL}(V)$ be a finite-dimensional real or complex representation of G_0 . If h_0 is a subalgebra of g_0 and W is an h_0 -invariant linear subspace of V then W is also $\bar{h}_0$ -invariant. In particular h_0 is an ideal of $\bar{h}_0$.

To prove the above statement let us consider the Grassmannian M of all linear subspaces of V of the same dimension as W with the action of G_0 on M defined by $g \cdot A = \tau(g)(A)$, $g \in G_0$, $A \in M$.

Given any $\varphi \in C^\infty(M)$ let $f \in C^\infty(G_0)$ be defined by $f(g) = \varphi(g \cdot W)$, $g \in G_0$. If $X \in g_0$ then

$$[Xf](g) = \left\{ \frac{d}{dt} \varphi(g \exp tX \cdot W) \right\}_{t=0}.$$

Thus if $X \in h_0$ then $Xf = 0$. Therefore $Xf = 0$ for all $X \in \bar{h}_0$. Hence $\varphi(\exp tX \cdot W) = \varphi(W)$ for all $t \in \mathbb{R}$, $X \in \bar{h}_0$, $\varphi \in C^\infty(M)$, which implies that W is $\bar{h}_0$ -invariant.

Finally if we consider the adjoint representation of G_0 then h_0 is an $\text{ad}(h_0)$ -invariant subspace of g_0 and therefore it is also $\text{ad}(\bar{h}_0)$ -invariant, which is equivalent to say that h_0 is an ideal of $\bar{h}_0$.

Example 4 (cf. [10, p. 615]). Let $\tau: G_0 \rightarrow \text{GL}(V)$ be a finite-dimensional complex representation of G_0 such that $d\tau$ is a faithful representation of g_0 . If

h_0 is a semisimple Lie subalgebra of g_0 then the corresponding connected Lie subgroup H_0 of G_0 is closed.

From the previous example, we know that h_0 is an ideal in $\bar{h}_0$, thus any element in $\bar{h}_0$ induces a derivation of h_0 . Since the only derivations of a semisimple Lie algebra are the inner ones we see that $\bar{h}_0 = h_0 + s_0$, where s_0 is the centralizer of h_0 in $\bar{h}_0$. Moreover the sum is direct because h_0 is centerless.

Let W be an h_0 -irreducible complex subspace of V . Then W is also $\bar{h}_0$ -invariant (cf. Example 3). From Schur's lemma we know that $d\tau(X)|_W$ is a scalar for all $X \in s_0$. Now take a projection P of V onto W commuting with $d\tau(h_0)$ and let $\tilde{\tau}: G_0 \rightarrow \text{End}(W)$ be defined by $P\tau(g)P = 0 \oplus \tilde{\tau}(g)$, $g \in G_0$, according to the decomposition $V = \ker P \oplus W$. If $f = \det \tilde{\tau}$ and $X \in \bar{h}_0$ we have

$$[Xf](g) = \left\{ \frac{d}{dt} \det(\tilde{\tau}(g)\tilde{\tau}(\exp tX)) \right\}_{t=0} = f(g) \text{tr}(d\tau(X)|_W).$$

Since h_0 coincides with its derived algebra it follows that $f \in C^\infty(G_0)^{h_0}$ and therefore $Xf = 0$ for all $X \in \bar{h}_0$, which implies that $\text{tr}(d\tau(X)|_W) = 0$ for all $X \in \bar{h}_0$. Thus $d\tau(X)|_W = 0$ if $X \in s_0$. Since any finite-dimensional representation of a semisimple Lie algebra is completely reducible, we obtain that $d\tau(X) = 0$, if $X \in s_0$, which in turn implies that $X = 0$. Therefore $\bar{h}_0 = h_0$ which is equivalent to H_0 being closed (Corollary 3).

As a corollary of this example we get the following results:

(i) If G_0 is a connected Lie group which is either simply connected or compact, and h_0 is a semisimple Lie subalgebra of g_0 then the corresponding connected Lie subgroup H_0 of G_0 is closed (Mostow [10, p. 615]).

(ii) If G_0 is a group of matrices and h_0 is a semisimple Lie subalgebra of g_0 then H_0 is closed (see also [4]).

Theorem 5. Let G_0 be a connected Lie group and let h_0 be a Lie subalgebra of the Lie algebra g_0 of G_0 . Then

$$\bar{h}_0 = \sum_{Y \in h_0} a_0(C^\infty(G_0)^Y),$$

where $C^\infty(G_0)^Y = \{f \in C^\infty(G_0) : Yf = 0\}$. Moreover there exists an abelian subalgebra v such that $\bar{h}_0 = h_0 \oplus v$.

Proof. We shall make use of the following result of Gotô [4, Theorem 1, p. 114]. Let H_0 be the connected Lie subgroup of G_0 with Lie algebra h_0 , let K be a maximal compact subgroup of $\bar{H}_0$, and let U be the radial of K . Then $\bar{H}_0 = UH_0$ and if we put $U_1 = U \cap H_0$ we have $\bar{U}_1 = U$.

Let u and u_1 be, respectively, the Lie algebras of U and U_1 . Then

$$\sum_{Y \in h_0} a_0(C^\infty(G_0)^Y) \supset h_0 + \sum_{Y \in u_1} a_0(C^\infty(G_0)^Y).$$

But since U is a torus and $\bar{U}_1 = U$ the well-known theorem of Kronecker implies that there exists $Y \in u_1$ such that $a_0(C^\infty(G_0)^Y) = u$. Hence

$$\sum_{Y \in h_0} a_0(C^\infty(G_0)^Y) \supset h_0 + u = \bar{h}_0.$$

Since the other inclusion is clear the first assertion is proved.

Finally by taking any direct complement v of u_1 in u the last statement follows.

Corollary 6 (Gotô [4]). *A necessary and sufficient condition for a connected Lie subgroup H_0 of G_0 to be closed is that H_0 contains the closure of every one-parameter subgroup of H_0 .*

2. THE HOLOMORPHIC SETTING

Since the problem under consideration involves a complex Lie subalgebra of the complexification $\mathfrak{g}$ of the Lie algebra of G_0 , it is reasonable to think of a complexification G of G_0 . But generally a Lie group has no complexification, unless it has a faithful finite-dimensional representation. A weaker demand is to look for an analytic homomorphism $\phi: G_0 \rightarrow G$ into a (connected) complex Lie group G such that $d\phi$ is injective. For G_0 a Lie group with a finite number of connected components, Hochschild and Mostow [7] have constructed a complex Lie group G and an analytic homomorphism $\phi: G_0 \rightarrow G$ such that for any finite-dimensional representation D of G_0 , there exists a unique holomorphic representation $\tilde{D}$ of G satisfying $D = \tilde{D} \circ \phi$. This is called the universal complexification of G_0 . Moreover it is known (cf. [11, Corollary to Proposition 10]) that if G_0 has a finite-dimensional representation τ such that $d\tau$ is faithful (for example if G_0 is semisimple), then the universal complexification $\phi: G_0 \rightarrow G$ of G_0 is such that $d\phi$ is injective and the Lie algebra $\mathfrak{g}$ of G coincides with the complexification of $d\phi(\mathfrak{g}_0)$. From now on we shall assume that $\phi: G_0 \rightarrow G$ satisfies these two conditions and we shall identify $\mathfrak{g}_0$ with $d\phi(\mathfrak{g}_0)$. Then for all $f \in \mathcal{O}(G)$ and $D \in \mathcal{U}(\mathfrak{g})$ we have

$$D(f \circ \phi) = (Df) \circ \phi.$$

We must be careful; this does not hold for all $f \in C^\infty(G)$. For example if $\phi: \mathbb{R} \rightarrow \mathbb{C}$ is the canonical injection and t and $z = x + iy$ are the usual coordinate functions on $\mathbb{R}$ and $\mathbb{C}$, respectively, then dt is identified with $\frac{\partial}{\partial x}$ and $i \frac{d}{dt}$ with $\frac{\partial}{\partial y}$. Now if we take $f(x, y) = y$ then $f \circ \phi \equiv 0$, thus $\frac{\partial}{\partial y}(f \circ \phi) \equiv 0$ while $(\frac{\partial}{\partial y} f) \circ \phi \equiv 1$.

In particular if $f \in \mathcal{O}(G)^h = \mathcal{O}(G) \cap C^\infty(G)^h$ then $f \circ \phi \in C^\infty(G_0)^h$. Also if $D \in \mathcal{A}(C^\infty(G_0)^h)$ then

$$[Df](e) = [D(f \circ \phi)](e) = 0$$

for all $f \in \mathcal{O}(G)^h$. But $\mathcal{O}(G)^h$ is stable under left multiplication by any $x \in G$, thus

$$[Df](x) = [D(f \circ L(x))](e) = 0$$

for all $f \in \mathcal{O}(G)^h$. Therefore we have

$$(4) \quad \mathcal{U}(\mathfrak{g})^h \subset \mathcal{A}(C^\infty(G_0)^h) \subset \mathcal{A}(\mathcal{O}(G)^h),$$

where $\mathcal{A}(\mathcal{O}(G)^h) = \{D \in \mathcal{U}(\mathfrak{g}) : Df = 0 \text{ for all } f \in \mathcal{O}(G)^h\}$.

Suppose H is a closed complex subgroup of G . We say that the homogeneous space $X = G/H$ has many holomorphic functions if there are holomorphic independent functions $f_1, \dots, f_n \in \mathcal{O}(X)$ (i.e., $df_1 \wedge \dots \wedge df_n \neq 0$), where $n = \dim_{\mathbb{C}} X$.

Theorem 7. Suppose h is the Lie algebra of a closed connected complex subgroup H of G . Then

$$\mathcal{A}(\mathcal{O}(G)^h) = \mathcal{U}(g)h$$

if and only if G/H has many holomorphic functions.

Proof. Let $m \subset g$ be a complex subspace such that $g = m \oplus h$, and let $X_1, \dots, X_p$ be a basis of g such that $X_i \in m$ for $i = 1, \dots, p$. If π denotes the natural mapping of G onto G/H , for a suitable open neighbourhood U of 0 in m the mapping

$$\pi(\exp(x_1 X_1 + \dots + x_p X_p)) \mapsto (x_1, \dots, x_p)$$

is a holomorphic coordinate system on the open neighbourhood $\pi(\exp U)$ of the origin o in G/H .

If $P \in S(g)$ is identified with a polynomial in the basis elements $X_1, \dots, X_p$ and $\partial_1, \dots, \partial_p$ denote the partial derivatives with respect to the complex variables $(x_1, \dots, x_p)$ then (cf. (2))

$$[\lambda(P)f](g) = [P(\partial_1, \dots, \partial_p)f(g \exp(x_1 X_1 + \dots + x_p X_p))](0)$$

for each $g \in G$, $f \in \mathcal{O}(G)$.

Let us assume now that G/H has many holomorphic functions. Then there exist $y_1, \dots, y_n \in \mathcal{O}(G/H)$ such that the map $x \mapsto (y_1(x), \dots, y_n(x))$ defines a holomorphic coordinate system in a neighbourhood V of the origin and such that $y_j(o) = 0$ for $j = 1, \dots, n$. Thus given $P \in S(m)$, $P \neq 0$, there exists $P' \in S(m)$, $P' \neq 0$ such that

$$\begin{aligned} & [P(\partial_1, \dots, \partial_n)f(\pi(\exp(x_1 X_1 + \dots + x_n X_n)))](0) \\ &= \left[P' \left(\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \right) f^*(y_1, \dots, y_n) \right](0) \end{aligned}$$

for all $f \in \mathcal{O}(G/H)$, where $f^*(y_1(x), \dots, y_n(x)) = f(x)$ for all $x \in V$. Now it is clear that we may choose $f \in \mathcal{O}(G/H)$ such that

$$[\lambda(P)(f \circ \pi)](e) \neq 0.$$

Since $\mathcal{U}(g) = \mathcal{U}(g)h \oplus \lambda(S(m))$ (see (1)) and $f \circ \pi \in \mathcal{O}(G)^h$ it follows that $\mathcal{A}(\mathcal{O}(G)^h) = \mathcal{U}(g)h$.

Conversely, let us assume that $\mathcal{A}(\mathcal{O}(G)^h) = \mathcal{U}(g)h$. Then making the natural identification between m and the holomorphic tangent space of G/H at o we know that

$$\{X \in m : df_o(X) = 0 \text{ for all } f \in \mathcal{O}(G/H)\} = \{0\}.$$

This is equivalent to say that the linear map $f \mapsto df_o$ from $\mathcal{O}(G/H)$ into m^* is surjective. Thus if $X_1, \dots, X_n$ is a basis of m there exist $f_1, \dots, f_n \in \mathcal{O}(G/H)$ such that $d(f_i)_o(X_j) = \delta_{ij}$, $i, j = 1, \dots, n$. This completes the proof that G/H has many holomorphic functions.

In those cases where $\mathcal{U}(g)h = A(\mathcal{O}(g)^h)$ we certainly have (see (4)) $\mathcal{U}(g)h = A(C^\infty(G_0)^h)$, but the converse need not be true. To see this let us consider the following example.

Example 5. Let G_0 be a connected noncompact semisimple Lie group and let K_0 be a maximal compact subgroup of G_0 . Assume that G_0/K_0 is an Hermitian symmetric space. If $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is the Cartan decomposition of $\mathfrak{g}_0$ corresponding to K_0 and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the corresponding complexification then $\mathfrak{p}$ can be identified with the complex tangent space of G_0/K_0 at the origin o . Let $\mathfrak{p}_+$ and $\mathfrak{p}_-$ be the linear subspaces of $\mathfrak{p}$ which correspond, respectively, to the holomorphic and antiholomorphic tangent spaces of G_0/K_0 at o . Thus it is known (cf. [5, Chapter VIII, Section 7]) that $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are abelian subalgebras of $\mathfrak{g}$. Also if G denotes the simply connected Lie group with Lie algebra $\mathfrak{g}$ and K, P_- are the analytic subgroups of G corresponding to the subalgebras $\mathfrak{k}$ and $\mathfrak{p}_-$, respectively, then $H = KP_-$ is a closed subgroup of G and G_0/K_0 can be holomorphically embedded as an open subset of G/H . Moreover, G/H is compact.

Now $C^\infty(G_0)^h = \{f \circ \pi : f \in \mathcal{O}(G_0/K_0)\}$. If $D \in \mathcal{A}(C^\infty(G_0)^h)$ then according to (1) $D = D_1 + D_2$ where $D_1 \in \mathcal{U}(\mathfrak{g})h$ and $D_2 \in \mathcal{U}(\mathfrak{p}_+)$. But $D_2(f \circ \pi) = 0$ for all $f \in \mathcal{O}(G_0/K_0)$ clearly implies $D_2 = 0$ since G_0/K_0 has many holomorphic functions. Thus

$$\mathcal{A}(C^\infty(G_0)^h) = \mathcal{U}(\mathfrak{g})h.$$

On the other hand $\mathcal{O}(G)^h = \mathbb{C}$ since G/H is compact and connected. Therefore we have $\mathcal{A}(\mathcal{O}(G)^h) = \mathcal{U}(\mathfrak{g})g$.

The complex manifold $X = G/H$ is said to be *quasi-affine* if it can be equivariantly embedded as a Zariski open set in an affine variety Y . If G is a connected complex reductive Lie group and G/H has many holomorphic functions then it can be proved that X carries a quasi-affine structure (see [9, p. 160]). Thus we have

Corollary 8. *Suppose G is a connected complex reductive Lie group and H a closed connected complex Lie subgroup with Lie algebra $\mathfrak{h}$. Then $\mathcal{A}(\mathcal{O}(G)^h) = \mathcal{U}(\mathfrak{g})h$ if and only if G/H is a quasi-affine variety.*

Results on when G/H carries a quasi-affine structure can be found in [1, 6]. If G is a connected complex reductive linear Lie group then it has one and only one structure of an affine algebraic group that is compatible with its structure as a complex Lie group. Moreover if H is a connected complex reductive Lie subgroup of G (H is necessarily closed in G) then G/H can be equipped with affine structures compatible with its structure of complex manifold (see [6, Theorem 3.2]). Therefore we can state

Corollary 9. *Let G be a connected complex reductive linear Lie group and let H be a connected complex reductive Lie subgroup with Lie algebra $\mathfrak{h}$. Then $\mathcal{A}(\mathcal{O}(G)^h) = \mathcal{U}(\mathfrak{g})h$.*

If G is an (affine) algebraic group over $\mathbb{C}$ and H is a unipotent algebraic subgroup of G then, by a theorem of Chevalley (see [8, Theorem 11.2]), there is a rational representation $\varphi: G \rightarrow \mathrm{GL}(V)$ and a one-dimensional subspace L of V such that

$$H = \{x \in G : \varphi(x)L = L\}.$$

Since a unipotent algebraic group has no nontrivial one-dimensional representation, if we pick $v \in L$, $v \neq 0$, we have

$$H = \{x \in G : \varphi(x)v = v\}.$$

In this way we see that the homogeneous space G/H can be identified with the orbit in V of the point v . Such an orbit is a quasi-affine variety (being open in its closure). Thus we have

Corollary 10. *If G is a linear algebraic group over $\mathbb{C}$ and H is a connected unipotent algebraic subgroup with Lie algebra h , then $\mathcal{A}(\mathcal{O}(G)^h) = \mathcal{U}(g)h$.*

If a connected complex reductive Lie group G acts holomorphically on X , then a theorem of Harish-Chandra (cf. [12, Theorems 4.4.2.1 and 4.4.3.1]) guarantees that the subspace of all holomorphic G -finite functions on X is a dense subset of the Fréchet space $\mathcal{O}(X)$. In fact, since G is the complexification of its maximal compact subgroup K (uniquely determined up to conjugacy) the subspace of all holomorphic G -finite functions on X coincides with the subspace of all holomorphic K -finite functions on X . Also if $\widehat{G}$ denotes the set of all equivalence classes of holomorphic irreducible finite-dimensional representations of G and if V_γ denotes a fixed G -module in the class $\gamma \in \widehat{G}$, then it is known that the subspace of all holomorphic G -finite functions (under the left regular representation) on G can be identified with $\bigoplus_{\gamma \in \widehat{G}} V_\gamma \otimes V_{\gamma^*}$. If h is a Lie subalgebra of the Lie algebra of G , V_γ^h will denote the subspace of all h -invariants in V_γ .

Theorem 11. *Let G be a connected complex reductive Lie group and let H be a connected closed complex Lie subgroup of G with Lie algebra h . Then*

$$\mathcal{U}(g)h = \{D \in \mathcal{U}(g) : D(V_\gamma^h) = 0 \text{ for all } \gamma \in \widehat{G}\}$$

if and only if G/H is a quasi-affine variety.

Proof. The linear map $f \mapsto f \circ \pi$ is a continuous G -morphism from $\mathcal{O}(G/H)$ onto $\mathcal{O}(G)^h$. Therefore $\bigoplus_{\gamma \in \widehat{G}} V_\gamma \otimes V_{\gamma^*}^h$ is dense in $\mathcal{O}(G)^h$. Since any $D \in \mathcal{U}(g)$ defines a continuous endomorphism of $\mathcal{O}(G)$ it follows that

$$\{D \in \mathcal{U}(g) : D(V_\gamma^h) = 0 \text{ for all } \gamma \in \widehat{G}\} = \mathcal{A}(\mathcal{O}(G)^h).$$

Now the theorem follows from Corollary 8.

Let g be a finite-dimensional complex semisimple Lie algebra and let $g = u \oplus h \oplus u_-$ be a triangular decomposition of g . Let $\widehat{g}$ denote the space of all finite-dimensional irreducible representations of g . If $\gamma \in \widehat{g}$ let V_γ denote a fixed g -module in the class γ and let V_γ^u denote the subspace of V_γ spanned by a highest weight vector in V_γ . A direct consequence of the above theorem is the following undoubtedly known result.

Corollary 12. *Let g be a finite-dimensional complex semisimple Lie algebra and let $g = u \oplus h \oplus u_-$ be a triangular decomposition of g . Then*

$$\mathcal{U}(g)u = \{D \in \mathcal{U}(g) : D(V_\gamma^u) = 0 \text{ for all } \gamma \in \widehat{g}\}.$$

Proof. Let G be a simply connected Lie group with Lie algebra g and let U be the connected Lie subgroup of G with Lie algebra u . Then G is a linear algebraic group and U is a maximal unipotent subgroup of G . Now the corollary follows from Theorem 11 since G/U is a quasi-affine variety.

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