THE DISCRETE SCHWARZ-PICK LEMMA FOR OVERLAPPING CIRCLES

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ABSTRACT. Let P and P' be circle packings in the hyperbolic plane such that they are combinatorically equivalent, neighboring circles in P overlap one another at some fixed angle between 0 and $\pi/2$ and the corresponding pairs of circles in P' overlap at the same angle, and the radius for any boundary circle of P is less than or equal to that of the corresponding boundary circle of P' . In this paper we show that the radius of any interior circle of P is less than or equal to that of the corresponding circle in P' , and the hyperbolic distance between the centers of circles in P is less than or equal to the distance between the corresponding circles in P' . Furthermore, a single instance of finite equality in either of the above implies equality for all.

1. INTRODUCTION

In a paper titled *The Schwarz-Pick Lemma for circle packings*, Beardon and Stephenson [1] prove what they consider a discrete analog for circle packings of the classical Schwarz-Pick Lemma. This result, termed the *Discrete Schwarz-Pick Lemma* (DSPL), has been useful in proving other discrete results (e.g., the uniqueness of infinite circle packings of the hyperbolic plane). But the DSPL is restricted in that it only considers circle packings in which neighboring circles are tangent to one another. Because circle packings in which neighboring circles overlap one another at an angle between 0 and $\pi/2$ (Figure 1 on the next page) have held (see [2]) and continue to hold promise for use in future work, we want to extend the DSPL to include these packings.

It will be assumed that the reader is familiar with the definitions and ideas in [1] and [2]. Many of the proofs given by Beardon and Stephenson in [1] can be extended without much difficulty for the generalized results of this paper. Consequently, many of the proofs and details are left to the reader.

2. PRELIMINARIES

Abstract 2-dimensional simplicial complexes are used to represent the combinatorics of a circle packing. We will restrict our attention to complexes, K , that are isomorphic to a finite triangulation of a closed topological disc and have

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FIGURE 1

the following properties: every boundary vertex shares an edge with at least one interior vertex, and any pair of interior vertices can be connected by an edge path whose edges have only interior vertices as end points. Complexes of this type will be termed *special*.

Let V and E be the set of vertices and edges of K respectively. A function $R: V \rightarrow (0, \infty]$ will be called a *label* for K . It will be assumed that R is finite on interior vertices. A function $\varphi: E \rightarrow [0, \pi/2]$ will be called an *angle assignment* for K if it satisfies the following:

- (i) if $e_1, e_2, e_3 \in E$ are distinct and form a closed edge path and $\varphi(e_1) + \varphi(e_2) + \varphi(e_3) \geq \pi$, then e_1, e_2, e_3 are the edges of a face of K ,
- (ii) if e_1, e_2, e_3, e_4 are distinct and form a closed edge path and $\varphi(e_i) = \pi/2$, $i = 1, 2, 3, 4$, then e_1, e_2, e_3, e_4 are the edges of two faces that share a common edge e .

A hyperbolic structure can be placed onto the complex K in the following manner. Let R and φ be given. Then to each face of K there corresponds a triple of radii and overlap angles. From these triples come three circles in the hyperbolic plane having the given radii and overlap angles. The hyperbolic triangle formed by joining the centers of these circles is associated to the face of K pointwise so that corresponding vertices are identified, thus inducing a hyperbolic structure onto this face. If two faces are contiguous, then their associated triangles can be made to be contiguous. By adjusting the identifications of those two faces, the hyperbolic structure is made compatible across the shared edge. The complex K along with the hyperbolic structure determined by R and φ is called a *hyperbolic complex* and will be denoted by $K(R, \varphi)$.

With the hyperbolic structure on K , each face of K has the structure of a hyperbolic triangle. At a given vertex of K , every face in the star of the given vertex forms an angle at this vertex. The sum of all these angles will be termed the *angle sum* at the vertex. We will call $K(R, \varphi)$ a *packing* if the angle sum

at every interior vertex is equal to 2π . $K(R, \varphi)$ will be called a *subpacking* if the angle sum at every interior vertex is greater than or equal to 2π .

The hyperbolic structure on $K(R, \varphi)$ is compatible across shared edges, but it may or may not be at shared vertices. If φ is constantly 0, there is a discussion in [1] of when the hyperbolic structure on $K(R, \varphi)$ is compatible at shared vertices and how it compares both locally and globally to the hyperbolic plane. This discussion can be extended to include any angle assignment φ .

3. MONOTONICITY RESULTS

We need to establish a collection of monotonicity results that will be essential in the proof of our main result. Refer to Figure 2 for

Lemma 1. *Let $r_0, r_1, r_2 \in (0, \infty]$ be the radii for the circles with centers z_0, z_1, z_2 and overlap angles $\alpha_{01}, \alpha_{02}, \alpha_{12} \in [0, \pi/2]$. Then:*

- (i) $\alpha(r_0)$ is a continuous strictly decreasing function.
- (ii) $\alpha(r_0)$ takes on every value in $[0, \pi - \alpha_{1,2})$ with $\alpha(r_0) \rightarrow \pi - \alpha_{12}$ as $r_0 \rightarrow 0$.
- (iii) $\beta(r_0), (\gamma(r_0))$ is a continuous strictly increasing function when $r_1 < \infty$ ($r_2 < \infty$), except if $\alpha_{01} = 0$ and $\alpha_{02} = \alpha_{12} = \pi/2$, ($\alpha_{02} = 0, \alpha_{01} = \alpha_{12} = \pi/2$), in which case it is constant.
- (iv) The area of the triangle of centers is a strictly increasing function of r_0 .
- (v) Let p be a point on the edge of the triangle of centers opposite z_0 . Let $l_P(r_0)$ denote the distance from p to z_0 . If at most one of r_1, r_2 is infinite then dl_P/dr_0 is positive and continuous.

Furthermore, if both r_1 and r_2 are finite then there exists a continuous function, $\tau(r_0)$, such that for any p , $dl_P/dr_0 > \tau(r_0) > 0$.

FIGURE 2. Triangle of centers and notation

An angle assignment φ will be called a *strict angle assignment* if $\varphi(e) \neq 0$ for any edge e such that e is the common edge of two contiguous faces where φ is $\pi/2$ on the remaining four edges. From Lemma 1 we get

Lemma 2. *Let ν be a vertex of $K(R, \varphi)$ with neighboring vertices $\nu_1, \dots, \nu_n$. Let $r, 0 < r < \infty$, and $r_1, \dots, r_n, 0 < r_i \leq \infty$, denote their radii respectively. Let $\theta = \theta(r, r_1, \dots, r_n)$ be the resulting angle sum at ν . Then:*

- (i) θ is continuous and strictly decreasing in r .
- (ii) θ is continuous and increasing in $r_i, 1 \leq i \leq n$.
- (iii) Fix $r_1, \dots, r_n$. Given any $\omega, 0 \leq \omega < \sum(\pi - \alpha_{ij})$, there exist a unique radius $r' > 0$ for which $\theta(r', r_1, \dots, r_n) = \omega$.

Furthermore, if φ is a strict angle assignment then monotonicity is strict in (ii).

Given R and φ , define the metric, $\rho_R(\cdot, \cdot)$, on $K(R, \varphi)$ to be the infimum of the lengths of paths between the two points in $K(R, \varphi)$. The notation $\rho_R(\cdot, \cdot)$ does not show that this metric is dependent on φ . This is because interest lies in what happens to $\rho_R(\cdot, \cdot)$ as the label is changed and φ remains fixed.

Lemma 3. *Let φ be fixed. If $K(R, \varphi)$ and $K(R', \varphi)$ are hyperbolic complexes such that $R'(\nu) \leq R(\nu)$ for every vertex ν then $\rho_{R'}(\nu_1, \nu_2) \leq \rho_R(\nu_1, \nu_2)$ for every pair of vertices. Also if ν_1 satisfies $R'(\nu_1) < R(\nu_1)$ and $\rho_R(\nu_1, \nu_2) < \infty$ then the inequality is strict.*

The proof consists of starting with a path in $K(R, \varphi)$ between ν_1 and ν_2 and showing the existence of a shorter path in $K(R', \varphi)$. This is accomplished by taking advantage of Lemma 1(v).

We want to find an upper bound on the radius at an interior vertex ν of $K(R, \varphi)$ with neighbors $\nu_1, \dots, \nu_n$. To do this, observe (refer to Figure 2)

- (i) If $0 < r_0 < \infty$ and $0 < r_1, r_2 \leq \infty$ then α is a decreasing function of α_{12} on $[0, \pi/2]$.
- (ii) If $0 < r_0 < \infty$ and $0 < r_1, r_2 \leq \infty$ then γ is an increasing function of α_{12} on $[0, \pi/2]$.
- (iii) When $\alpha_{01} = \alpha_{02} = \pi/2$ and $\alpha_{12} = 0$, the circle centered at z_0 intersects the other two circles at their point of tangency.

Lemma 4. *Let ν be an interior vertex of $K(R, \varphi)$ with neighbors $\nu_1, \dots, \nu_n$. Let $R(\nu) = r < \infty$ and $R(\nu_i) = r_i$. If the angle sum $\theta(\nu)$ at ν is greater than or equal to 2π then $r < \sqrt[n]{n}$.*

Proof. By observations (i) and (ii), decreasing the overlap angles between all pairs of neighboring circles having vertices in $\{\nu_1, \dots, \nu_n\}$ to zero and increasing the overlap angles between the pairs of circles having vertices ν_i and ν to $\pi/2$ will increase θ . By Lemma 2 increasing each r_i to ∞ will again increase θ . By Lemma 2(iii) r can be increased to some r' until $\theta = 2\pi$. Let P be the polygon formed by connecting neighboring ν_i 's with geodesics. By observation (iii) the circle of radius r' is contained in P . Comparing the area of the circle with that of the polygon, we have that

$$\pi(r')^2 \leq 4\pi(r'/2)^2 \leq 4\pi \sinh^2(r'/2) \leq (n-2)\pi. \quad \square$$

5. THE DISCRETE SCHWARZ-PICK LEMMA FOR OVERLAPPING CIRCLES

Let K , φ , and boundary radii for K be given. Define $\mathfrak{R}$ to be the set of all labels R such that $K(R, \varphi)$ is a subpacking and has boundary radii less than or equal to the given boundary radii. By the Andreev-Thurston Theorem we have that $\mathfrak{R}$ is nonempty. Define $\sup \mathfrak{R}(v) = \sup\{R(v) : R \in \mathfrak{R}\}$ for every vertex v of K . Using the monotonicity results of Lemma 2 and the bound for interior radii given in Lemma 4, it is not hard to see that $\sup \mathfrak{R}$ is the unique packing label for K having the prescribed boundary radii. We are now in a position to prove a generalization of the DSPL.

Theorem 1. *Let K be a special complex and let φ be an angle assignment for K . If R and R_P are labels for K such that $K(R, \varphi)$ is a subpacking, $K(R_P, \varphi)$ is a packing, and $R(v) \leq R_P(v)$ for every boundary vertex v , then*

- (i) $R(v) \leq R_P(v)$ for every vertex v ,
- (ii) $\rho_R(v_1, v_2) \leq \rho_{R_P}(v_1, v_2)$ for any pair of vertices in K , and
- (iii) $\text{Area}_R(f) \leq \text{Area}_{R_P}(f)$ for every face f of K .

Furthermore, if φ is a strict angle assignment and equality holds for either a single interior vertex in (i), a single pair of interior vertices in (ii), or a single face in (iii) then $R = R_P$.

Proof. (i) The previous discussion gives this result. If φ is a strict angle assignment and equality holds at a single interior vertex v_0 then we claim it holds for all the neighboring vertices. Suppose that there exists a neighbor w such that $R(w) < R_P(w)$. Since the angle sum at v_0 is an increasing function of the radius at w and $\theta_R(v_0) \geq 2\pi$, $\theta_{R_P}(v_0) > 2\pi$. This contradicts that R_P is a packing label. Hence there must be equality at all of v_0 's neighbors. This equality spreads to all vertices by repeating the above argument.

(ii) Lemma 3 and (i) give this result. The case of equality comes from the last part of Lemma 3 and (i).

(iii) Every face has at least one interior vertex v . This and Lemma 1(iv) give the inequality. Since monotonicity is strict, the area of two faces being equal implies $R(v) = R_P(v)$. By (i), $R = R_P$. $\square$

Simple examples show that φ needs to be a strict angle assignment in the case of equality.

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