A COUNTEREXAMPLE CONCERNING THE MAXIMUM AND MINIMUM OF A SUBHARMONIC FUNCTION

ALEXANDER FRYNTOV

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ABSTRACT. For every $\Delta > 0$ a function u subharmonic in the plane is constructed such that u has the order $\rho = 1 + \Delta$ and satisfies the condition

$$\min_{\vartheta} u(re^{i\vartheta}) / \max_{\vartheta} u(re^{i\vartheta}) \leq -(C + 1) \quad \text{for every } r > 0,$$

where $C = C(\rho) > 0$. This example answers a question of W. K. Hayman.

Let f be an entire function of finite order ρ , and let

$$M(r, f) = \max_{\vartheta} |f(re^{i\vartheta})|, \quad \mu(r, f) = \min_{\vartheta} |f(re^{i\vartheta})|.$$

The well-known $\cos \pi \rho$ -theorem asserts that if $\rho \leq 1$, then

$$\limsup_{r \rightarrow \infty} \frac{\log \mu(r, f)}{\log M(r, f)} \geq \cos \pi \rho.$$

In 1952 Hayman [H2] proved that for every $\rho > \rho_0$, ρ being positive and large enough, there exists an entire function of order ρ such that

$$\limsup_{r \rightarrow \infty} \frac{\log \mu(r, f)}{\log M(r, f)} < -c(\rho) < -1;$$

in fact, $c(\rho)$ is unbounded as $\rho \rightarrow \infty$. However, the question [H1] of what happens when ρ is close to one remained open up to the present time. The purpose of this paper is to give a particular answer to this question.

In fact, for every $\rho > 1$ we construct a function u subharmonic in $\mathbb{C}$ that satisfies the conditions:

- (i) $u(0) = 0$;
- (ii) $B(r, u) = r^\rho$, $0 < r < \infty$;
- (iii) $L(u, r)/B(u, r) < -(1 + C(\rho))$ for every $r > 0$ and some $C(\rho) > 0$.

(Here L and B denote the lower and upper bounds of $u(z)$ on the circle $|z| = r$.)

In order to get a corresponding example of an *entire* function, it is sufficient to use a suitable theorem on the approximation of a subharmonic function by

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the logarithm of modulus of an entire one (see [A] or [Yu]). The details will be given in Appendix 2.

TWO LEMMAS

To construct this example we need two lemmas.

Lemma 1. *For every $T > 1$ and every τ , $T > \tau > 1$, there exists a subharmonic function g such that for some $\alpha > 0$ the following relations are valid:*

$$(1) \quad B(g, r) = g(-r) = r,$$

$$(2) \quad g(r) < -(1 + \alpha)r \quad \forall r \in \bigcup_{k=-\infty}^{\infty} [T^k, \tau T^k] = \mathcal{R}^+(T, \tau).$$

Proof. First we introduce some notation. Let T, τ be positive numbers such that $1 < \tau < T$, and let I_ϵ be the complex ϵ -neighborhood of the interval $I = I_0 = [1, \tau]$. Let

$$\mathcal{R}_\epsilon(T, \tau) = \bigcup_{k=-\infty}^{\infty} T^k I_\epsilon.$$

Denote by $\mathcal{U}_\epsilon$ the family of subharmonic functions that satisfy the inequality ($z = re^{i\theta}$)

$$(3) \quad u_\epsilon(re^{i\theta}) \leq \begin{cases} r(1 + \cos \theta) & \text{for } z \notin \mathcal{R}_\epsilon(T, \tau), \\ 0 & \text{for } z \in \mathcal{R}_\epsilon(T, \tau). \end{cases}$$

Let u be the least upper bound of this family. Obviously, u is a subharmonic function. This function satisfies the relations

$$(4) \quad u(Tz) = Tu(z)$$

and

$$(5) \quad u(-r) = 0 \quad \forall r \geq 0.$$

If $u(z) < r(1 + \cos \theta)$ then u is harmonic in some δ -neighborhood of z ; indeed, otherwise the function u_δ continuous in $\mathbb{C}$, coinciding with u outside this neighborhood and harmonic inside, belongs to $\mathcal{U}_\epsilon$. The same argument shows that if $u \not\equiv 0$ then $u > 0$ outside $(-\infty, 0] \cup \mathcal{R}_\epsilon(T, \tau)$. We will prove later that $u \not\equiv 0$. Assuming this let us finish the proof of the lemma. If $\nu > 0$ is small enough then

$$u(z) < r(1 + \cos \theta), \quad z = re^{i\theta} \in I_{\epsilon+\nu},$$

and, hence, u is harmonic in $I_{\epsilon+\nu} \setminus \bar{I}_\epsilon$. Since u is positive and harmonic in $I_{\epsilon+\nu} \setminus \bar{I}_\epsilon$ and $u(z) = 0$, $z \in \bar{I}_\epsilon$, the normal derivative of u (in the direction of outward normal to the boundary of $\bar{I}_\epsilon$) has positive infimum on $\partial \bar{I}_\epsilon$. Denote by $G_{I_\epsilon}(z, 1)$ the Green function of I_ϵ with the pole at 1. We remark that its normal derivative is bounded since the boundary of I_ϵ is smooth. Hence, the function

$$v(z) = \begin{cases} u(z) & \text{for } z \in \mathbb{C} \setminus \mathcal{R}_\epsilon(T, \tau), \\ -\gamma T^k G_{I_\epsilon}(z/T^k, 1) & \text{for } z \in T^k I_\epsilon \end{cases}$$

is subharmonic if $\gamma > 0$ is sufficiently small. It is clear that

$$(6) \quad v(r) \leq -\alpha r, \quad r \in \mathcal{R}(T^+, \tau),$$

for some positive α .

The function

$$g(re^{i\theta}) = -r \cos \theta + v(re^{i\theta})$$

satisfies all conditions of Lemma 1 due to properties (3)–(6).

It remains to prove that $u \not\equiv 0$ with an appropriate choice of ϵ . Consider the domains

$$\Omega_{\epsilon, \delta} = \{z : |\arg z| < \pi - \delta, z \notin \mathcal{R}_\epsilon^+(T, \tau)\}$$

for some $\delta \geq 0$.

There exists a function $u_{\epsilon, \delta}$ positive and harmonic inside the domain $\Omega_{\epsilon, \delta}$ and vanishing on its boundary. This function is unique up to a multiplicative constant (see Appendix 1). As the region $\Omega_{\epsilon, \delta}$ is invariant under the transformation $z \mapsto Tz$, we conclude from the uniqueness of $u_{\epsilon, \delta}$ that

$$(7) \quad u_{\epsilon, \delta}(Tz) = T^\rho u_{\epsilon, \delta}(z)$$

for some positive $\rho = \rho(\epsilon, \delta)$. We always extend $u_{\epsilon, \delta}$ to a subharmonic function in the plane by putting it equal to zero outside $\Omega_{\epsilon, \delta}$. To complete the proof we need to show that for every fixed T and $\tau \in (1, T)$ there exist ϵ and δ such that

$$(8) \quad \rho(\epsilon, \delta) = 1.$$

In this case the function $u_{\epsilon, \delta}(z)$ after multiplication by a suitable positive constant will belong to $\mathcal{U}_\epsilon$.

To prove (8) we show that $\rho(\epsilon, \delta)$ is continuous for all admissible (so small that $\Omega_{\epsilon, \delta}$ is connected) values of ϵ and δ and takes values greater and smaller than 1.

First, we show that $\rho(0, 0) < 1$. Indeed, $u_{0,0}$ is positive and harmonic in the upper halfplane. So

$$(9) \quad \int_{\mathbf{R}} \frac{u_{0,0}(x)}{1+x^2} dx < \infty,$$

which is consistent with (7) only if $\rho < 1$.

Second, it follows from the Phragmén-Lindelöf Principle that $\rho(\epsilon, \delta) > 1$ if $\delta > \pi/2$.

Since the continuity of $\rho(\epsilon, \delta)$ is proved in Appendix 1, this completes the proof of Lemma 1.

We need also the following lemma.

Lemma 2. *For every $\Delta \in (0, 1)$ and $T > 1$ there exist $\tau \in (1, T)$ and a subharmonic function v such that*

$$(10) \quad v(-r) = B(v, r) = r^\Delta \quad \text{for } r \in [0, \infty),$$

$$(11) \quad v(r) < -2r^\Delta \quad \text{for } r \in \bigcup_{k=-\infty}^{\infty} [\tau^{-1/\Delta} T^{k/\Delta}, T^{k/\Delta}].$$

Proof. Let $M = T^{1/\Delta}$. Consider the function

$$(12) \quad g(z) = \int_0^{+\infty} \log \left| 1 - \frac{z^2}{t^2} \right| d\mu(t),$$

where μ is the measure supported by the set $\{M^k\}_{k=-\infty}^{\infty}$ and

$$\mu\{M^k\} = M^{\Delta k}, \quad k \in \mathbb{N}.$$

It is easy to see that integral (12) defines a subharmonic function with $g(Mz) = M^\Delta g(z)$, $z \in \mathbb{C}$. Now put

$$f(re^{i\theta}) = f(re^{-i\theta}) = r^\Delta \cos(\Delta(\theta - \pi/2)), \quad 0 \leq \theta \leq \pi.$$

The function f is subharmonic in the plane. Then the function

$$u_\gamma(z) = \gamma g(z) + f(z)$$

is subharmonic for every positive γ and satisfies the relations

$$(13) \quad r^\Delta(1 + \gamma A) \geq u_\gamma(\pm ir) = B(r, u_\gamma) \geq r^\Delta(1 - \gamma A), \quad r \in \mathbb{R},$$

$$(14) \quad u_\gamma(Mz) = M^\Delta u_\gamma(z), \quad z \in \mathbb{C},$$

where $A = \sup\{g(ir)/r^\Delta : r > 0\}$ (it is clear that $0 < A < \infty$). Setting

$$\tilde{u}_\gamma(re^{i\theta}) = \begin{cases} u_\gamma(z) & \text{for } |\theta| \leq \pi/2, \\ \max\{B(r, u_\gamma), (1 + \gamma A)r^\Delta \cos(\Delta(\pi - \theta))\} & \text{for } |\pi - \theta| < \pi/2, \end{cases}$$

we show that according to (13) this function is subharmonic if $\gamma > 0$ is small enough. Indeed, if $\gamma > 0$ is sufficiently small then the inequality

$$(1 + \gamma A) \cos(\Delta\pi/2) < (1 - \gamma A) \leq B(r, u_\gamma)r^{-\Delta}$$

is valid, so $\tilde{u}_\gamma$ is equal to u_γ in some angle $\{re^{i\theta} : |\theta| < \delta\}$, $\delta > \pi/2$, and $\tilde{u}_\gamma(-r) = B(r, \tilde{u}_\gamma) = r^\Delta(1 + \gamma A)$.

Using (13), (14), and the equality $\tilde{u}_\gamma(1) = -\infty$ we conclude that the function

$$v(z) = (1 + \gamma A)^{-1} \tilde{u}_\gamma(z)$$

satisfies all conditions of Lemma 2 for some suitable τ . Lemma 2 is proved.

CONSTRUCTION OF THE EXAMPLE

Suppose $\rho \in (1, 2)$, and let $\Delta = \rho - 1$. We choose $T > 1$ arbitrarily; for example, put $T = 2$. According to Lemma 2 there exist $\tau \in (1, T)$ and a subharmonic function v satisfying (10) and (11). According to Lemma 1 for this τ there exists a subharmonic function g which satisfies (1) and (2) for some $\alpha > 0$. Define the function u in the following way:

$$u(re^{i\theta}) = \begin{cases} g(r^{1+\Delta}e^{i\theta(1+\Delta)}) & \text{for } |\theta| \leq \pi/(1+\Delta), \\ v(-r^{1+1/\Delta}e^{i\theta(1+1/\Delta)}) & \text{for } |\pi - \theta| < \pi\Delta/(1+\Delta). \end{cases}$$

It follows from (1) and (10) that this function u is subharmonic in the plane and satisfies (ii). Property (iii) follows from (2) and (11).

For $\rho \geq 2$ consider $u(z^n)$ with a suitable $n \in \mathbb{N}$.

The example is constructed.

APPENDIX 1

The results in this appendix are essentially known, but we could not find a convenient reference.

1. Let $\Omega = \Omega_{\epsilon, \delta}$ be the domain described in the proof of Lemma 1. We prove that there exists a function u positive and harmonic in Ω and vanishing on $\partial\Omega$. Denote by G the Green function of Ω ; fix a point $z_0 \in \Omega$; and choose a sequence $z_k \rightarrow \infty$, $z_k \in \Omega$. Then the sequence of positive harmonic functions $u_k(z) = G(z, z_k)/G(z_0, z_k)$ is equicontinuous at every point $z \in \Omega$. If we extend u_k by putting $u_k(z) = 0$, $z \in \mathbb{C} \setminus \Omega$, then the sequence u_k will be equicontinuous in $\bar{\Omega}$. This follows from a simple application of the two-constants theorem. Thus any limit function u of the sequence u_k is positive and harmonic in Ω and vanishes on the boundary. We have $u(z_0) = 1$, so $u \not\equiv 0$.

2. Let us prove that a positive harmonic function in Ω vanishing on the boundary is unique up to a multiplication by a positive constant. We use a modification of the argument in [Kj].

Let U be the set of all such functions. Without loss of generality we may suppose that δ in the definition of the domain Ω is equal to $\pi/2$ since the general case may be reduced to this by conformal mapping.

Let $K = \{r_0 e^{i\theta} : |\theta| \leq \pi/2\}$, $r_0 \in \Omega$. Let $u \in U$. It is easy to see that the function $g(z) = u(z)/\Re z$ defined in the right halfplane can be extended continuously to K and has a positive infimum on K .

The basic assertion that we will use for the proof of the uniqueness is the following. There exists $d > 0$ such that for every $u \in U$ we have

$$(15) \quad \frac{\max_{z \in K} g(z)}{\min_{z \in K} g(z)} \leq d.$$

Assume that (15) is false. Then there exists a sequence $u_k(z) \in U$ such that

$$(16) \quad \frac{\max_{z \in K} g(z)}{\min_{z \in K} g(z)} > k.$$

We may suppose $B(2r_0, u_k) = 1$. By the Harnack inequality the sequence $\{u_k\}$ is a precompact family of harmonic functions on Ω , and by normalization no subsequence of this sequence tends to ∞ or to identical 0. Applying the two-constants theorem we see that all limit functions vanish on the boundary, so all limit functions belong to U . So (16) cannot occur and (15) holds.

We remark that (15) remains true (with the same constant d) if we take $z \in T^k K$ instead of $z \in K$ since U is invariant with respect to the transformations $u(z) \mapsto u(T^k z)$.

Let $u, v \in U$ be two different functions. Let α_k be the sequence defined as

$$\alpha_k = \sup\{\alpha : u(z) - \alpha_k v(z) \geq 0 \text{ for } z \in T^k K\}.$$

It is evident that $\alpha_k \searrow \alpha$ for some nonnegative α . Then set $\eta_k = \alpha_k - \alpha \rightarrow 0$. The sequence

$$u(z) - \alpha_k v(z) = r_k(z)$$

converges to a nonnegative function $r(z) = r_k(z) + \eta_k v(z)$, and either $r(z) \in U$ or $r(z) \equiv 0$. According to the definition of α_k the function $r_k(z)/\Re z$ is zero

at some point of $T^k K$, since otherwise (15) implies that $(r_k(z) - \eta v_k(z))/\Re z$ will be positive on $T^k K$ for some $\eta > 0$. Thus we have

$$\frac{r(z_k)}{\Re z_k} = \eta_k \frac{v(z_k)}{\Re z_k}$$

for some point $z_k \in T^k K$. Therefore, applying (15) to both the left and right sides of the last relation we obtain

$$\frac{r(z)}{\Re z} < d^2 \eta_k \frac{v(z)}{\Re z}$$

for every point $z \in K$. (This proof is valid when $r(z) \in U$. When $r(z) \equiv 0$ this inequality is trivial.) By the maximum principle

$$r(z) < d^2 \eta_k v(z) \quad \text{for } |z| < T^k K;$$

hence $r(z) \equiv 0$. So we get $u(z) = \alpha v(z)$, and hence the uniqueness is proved.

3. Finally we prove that the order $\rho(\epsilon, \delta)$ of the positive harmonic function $u_{\epsilon, \delta}$ in $\Omega_{\epsilon, \delta}$ vanishing on the boundary is continuous for $0 \leq \delta \leq \pi/2$ and $0 \leq \epsilon \leq \epsilon_0$, where ϵ_0 is small enough. It follows from the uniqueness of $u_{\epsilon, \delta}$ that $u_{\epsilon, \delta}(Tz) = Cu_{\epsilon, \delta}(z)$, where $C = T^\rho$. So

$$(17) \quad \rho(\epsilon, \delta) = \frac{\log(u_{\epsilon, \delta}(Tz_0)/u_{\epsilon, \delta}(z_0))}{\log T},$$

where z_0 is an arbitrary point in $\Omega_{\epsilon, \delta}$. Assume that $\epsilon_n \rightarrow \epsilon_0$, $\delta_n \rightarrow \delta_0$. We may suppose that $z_0 \in \Omega_{\epsilon_n, \delta_n}$ for all n . Then (after selection of a subsequence if necessary) we have $u_{\epsilon_n, \delta_n} \rightarrow v$, on compacta in $\Omega_{\epsilon_0, \delta_0}$, where v is a positive harmonic function in $\Omega_{\epsilon_0, \delta_0}$, vanishing on the boundary. From the uniqueness it follows that $v = Cu_{\epsilon_0, \delta_0}$, and we have

$$\rho(\epsilon_n, \delta_n) \rightarrow \rho(\epsilon_0, \delta_0)$$

in view of (17).

APPENDIX 2

In this appendix we show how to get the corresponding entire example from the subharmonic one. We need an approximation theorem of Azarin [A] and the notion of a C_0 -set.

First we give the definition of a C_0 -set. A set E is called a C_0 -set if it may be covered by disks $B(\rho_k, \xi_k)$ (ρ_k are their radii, ξ_k are the centers) such that the relation

$$\lim_{r \rightarrow \infty} \frac{1}{r} \sum_{|\xi_k| \leq r} \rho_k = 0$$

is valid.

Now we cite Azarin's theorem on the approximation of subharmonic functions.

Theorem A. *Let $u(z)$ be a subharmonic function of finite proximate order $\rho(r)$. Then there exists an entire function $f(z)$ of the same proximate order $\rho(r)$ and the same type with respect to this order, and there exists a C_0 -set E such that the estimate*

$$\log |f(z)| - u(z) = o(|z|^{\rho(|z|)}), \quad |z| \rightarrow \infty,$$

holds outside E .

Let D be an open set intersecting every circle $|z| = r$ such that $TD = D$, T being taken from the definition of the subharmonic function $u(z)$. Let E be a C_0 -set. It is easy to see that there exists r_0 such that the set $D \setminus E$ intersects every circle of radii larger than r_0 . Indeed, if this is not so then there exists a sequence $r_k \rightarrow \infty$ such that

$$\text{mes}\{\varphi : r_k e^{i\varphi} \in E\} \geq \text{mes}\{\varphi : r_k e^{i\varphi} \in D\} \geq \delta > 0, \quad r_k \rightarrow \infty.$$

But the latter inequality contradicts the definition of a C_0 -set.

Now taking $\epsilon > 0$ small enough to satisfy $1 + \alpha - \epsilon > 1$ and setting $D = \{z : u(z) < -(1 + \alpha - \epsilon)|z|^\rho\}$ we can construct by Azarin's theorem an entire function $f(z)$ and a C_0 -set E such that

$$B(|z|, \log |f(z)|) - B(|z|, u(z)) = o(|z|^\rho), \quad |z| \rightarrow \infty, \quad z \in \mathbb{C},$$

and

$$\log |f(z)| - u(z) = o(|z|^\rho), \quad |z| \rightarrow \infty, \quad z \in \mathbb{C} \setminus D.$$

Since the set $D \setminus E$ intersects every circle $|z| = r$ for all sufficiently large $r > r_0$, we have

$$B(|z|, \log |f(z)|) = (1 + o(1))|z|^\rho, \quad |z| \rightarrow \infty,$$

and

$$L(|z|, \log |f(z)|) < -(1 + \alpha - \epsilon)(1 + o(1))|z|^\rho, \quad |z| \rightarrow \infty,$$

and hence the function $f(z)$ gives the needed example.

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DIVISION OF MATHEMATICS, INSTITUTE FOR LOW TEMPERATURE PHYSICS AND ENGINEERING,
47 LENIN AVENUE, KHARKOV 310164, UKRAINE
E-mail address: fryntov@ilt.kharkov.ua