EXPONENTIATION IS HARD TO AVOID

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ABSTRACT. Let $\mathcal{R}$ be an O-minimal expansion of the field of real numbers. If $\mathcal{R}$ is not polynomially bounded, then the exponential function is definable (without parameters) in $\mathcal{R}$. If $\mathcal{R}$ is polynomially bounded, then for every definable function $f : \mathbb{R} \rightarrow \mathbb{R}$, f not ultimately identically 0, there exist $c, r \in \mathbb{R}$, $c \neq 0$, such that $x \mapsto x^r : (0, +\infty) \rightarrow \mathbb{R}$ is definable in $\mathcal{R}$ and $\lim_{x \rightarrow +\infty} f(x)/x^r = c$.

In the following, let $\mathcal{R} := (\mathbb{R}, <, 0, 1, +, \cdot, \dots)$ be an expansion of the ordered field of real numbers. “Definable” means first-order definable in $\mathcal{R}$ with parameters from $\mathbb{R}$. A function $f : X \rightarrow \mathbb{R}$, $X \subseteq \mathbb{R}$, is said to be definable if its graph is definable. We may, whenever convenient, assume that we deal with totally defined functions by setting partial functions equal to 0 off their domain of definition. For the rest of this note, all functions mentioned are of one variable.

We say that $\mathcal{R}$ is *polynomially bounded* if, for every definable function f , there exists $N \in \mathbb{N}$ such that ultimately $|f(x)| \leq x^N$. (“Ultimately” abbreviates “for all sufficiently large positive x ”.) We say that $\mathcal{R}$ is *O-minimal* if the definable subsets of $\mathbb{R}$ are just the finite unions of intervals of all kinds, including singletons. (For general facts about O-minimal structures, see [PS] and [KPS].)

Given definable functions f and g such that ultimately $g(x) \neq 0$, we write $f \sim g$ if $\lim_{x \rightarrow +\infty} f(x)/g(x) = 1$.

Theorem. *Let $\mathcal{R}$ be O-minimal and not polynomially bounded. Then the exponential function is definable.*

Proof. By [vdD], every definable function is ultimately differentiable and either constant or strictly monotone. Thus, the germs at $+\infty$ of definable functions form a Hardy field. In particular, every definable function has a limit in $\mathbb{R} \cup \{-\infty, +\infty\}$ at $+\infty$. Since $\mathcal{R}$ is not polynomially bounded, there exists a definable function f such that, for all $n \in \mathbb{N}$, $\lim_{x \rightarrow +\infty} f(x)/x^n = +\infty$. By Proposition 6 of [R], there exists a definable function h with $h' \sim f'/f$. Since f is ultimately strictly increasing, f has a definable compositional inverse f^{-1} .

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for large positive x . Then $(h \circ f^{-1})' \sim 1/x$ (where x denotes the identity function). Let $g = h \circ f^{-1}$, so g is definable and $g' \sim 1/x$.

We now show that the existence of a definable function g with $g' \sim 1/x$ implies that $\log$ is definable.

First, we claim that, for all $t > 0$, $t \neq 1$, $\lim_{x \rightarrow +\infty} (g(tx) - g(x))$ is finite and nonzero.

Suppose $t > 1$. By the Mean Value Theorem, for large x we have

$$(g(xt) - g(x))/(t - 1) = xg'(\xi(x)), \quad \text{with } x < \xi(x) < xt.$$

Note then that $1/t < x/\xi(x) < 1$. Since $\lim_{x \rightarrow +\infty} xg'(x) = 1$, ultimately we have $|xg'(\xi(x)) - x/\xi(x)| < x/(2t\xi(x))$, and thus

$$|(g(xt) - g(x))/(t - 1) - x/\xi(x)| < 1/(2t).$$

So ultimately $(t - 1)/(2t) < g(xt) - g(x) < (t - 1)(1 + 1/(2t))$. Now $x \mapsto g(xt) - g(x)$ is definable, so $\lim_{x \rightarrow +\infty} (g(xt) - g(x))$ exists, and the claim holds.

For $t \in (0, 1)$, note that

$$\lim_{x \rightarrow +\infty} (g(xt) - g(x)) = - \lim_{x \rightarrow +\infty} (g(x/t) - g(x)).$$

We have now shown that there is a definable function $G : (0, +\infty) \rightarrow \mathbb{R}$ with $G(t) = \lim_{x \rightarrow +\infty} (g(xt) - g(x))$. Note that $G(1) = 0$, so G is nonconstant. It is easy to check that $G(st) = G(s) + G(t)$ for all $s, t > 0$. By O-minimality, G is ultimately differentiable. By elementary real analysis, we have $G(t) = G'(1) \log t$ for all $t > 0$. Since $G'(1) \neq 0$, $\log$ is definable, and thus $\exp$ is as well. $\square$

Remark. The exponential function is thus even definable without parameters from $\mathbb{R}$, using the fact that the exponential function is the unique differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$ with $f(0) = 1$ and $f' = f$; see [vdD].

The first known example of an O-minimal expansion of the ordered field of real numbers that is not polynomially bounded is due to Wilkie, who has recently shown (in [W1] and [W2]) that $(\mathbb{R}, <, 0, 1, +, \cdot, \exp)$ is O-minimal.

Proposition. *Let $\mathcal{R}$ be O-minimal and polynomially bounded. Let f be definable and not ultimately identically 0. Then there exist $c, r \in \mathbb{R}$, $c \neq 0$, such that $f \sim cx^r$ and $x \mapsto x^r : (0, +\infty) \rightarrow \mathbb{R}$ is definable.*

Proof. Since $\mathcal{R}$ is polynomially bounded, by the remarks immediately preceding Proposition 4 of [R], there exist c, r such that the desired asymptotic condition holds. Then $x \mapsto x^r : (0, +\infty) \rightarrow \mathbb{R}$ is definable (as noted in [P]), since, for all $x > 0$, $\lim_{y \rightarrow +\infty} f(xy)/f(y) = x^r$. $\square$

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