AN EXAMPLE OF A QUANTUM GROUP: THE TWISTED $\mathrm{Sp}_q(2)$

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ABSTRACT. We investigate the structure of the quantum symplectic group $\mathrm{Sp}(2)$ and give an explicit form for the fundamental representation. We prove that this group is a deformation of the classical compact group $\mathrm{Sp}(2)$ by showing that it is equivalent to the Reshetikhin, Takhtajan, Faddeev quantum $\mathrm{Sp}(2)$.

0. INTRODUCTION

Quantum groups have been introduced by Drinfeld [1], Jimbo [2], Reshetikhin, Takhtajan, and Faddeev [5, 9], and Woronowicz [3]. Drinfeld defined a quantum group as the spectrum of a Hopf algebra, while in Jimbo's approach quantum groups are deformations of the universal enveloping algebra of a semisimple Lie algebra. Reshetikhin, Takhtajan, and Faddeev [5, 9] define a formal quantum group as the associative algebra with operators as generators and relations. In their theory the basic commutation relations of operators is described by a solution of the Yang-Baxter equation.

From a different point of view Woronowicz considers a quantum group to be the deformation of the C^* -algebra of continuous functions on a compact group. He gives the explicit construction of the quantum groups $\mathrm{SU}(2)$ and $\mathrm{SU}(N)$ with $N \geq 2$.

In this note we use Woronowicz's approach to quantum groups to construct a quantum deformation of the symplectic group $\mathrm{Sp}(2)$. The quantum $\mathrm{Sp}(2)$ à la Woronowicz may also be viewed as a specialization of the Reshetikhin, Takhtajan, and Faddeev [5, 9] construction of quantum groups.

1. THE RESHETIKHIN, TAKHTAJAN, FADDEEV CONSTRUCTION

Let $V = \mathbb{C}^n$ be an n -dimensional $\mathbb{C}^n$ -vector space. Consider R belonging to $\mathrm{Mat}(V^{\otimes 2}, \mathbb{C})$ such that it satisfies the Yang-Baxter equation

$$(1.1) \quad R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

where the lower indices describe the embedding of the matrix R into $\mathrm{Mat}(V^{\otimes 3}, \mathbb{C})$. Let $A = A(R)$ be the associative algebra over $\mathbb{C}$ with generators $1, t_{i,j}, i, j = 1, \dots, n$, satisfying the relation

$$(1.2) \quad R(T \otimes I)(I \otimes T) = (I \otimes T)(T \otimes I)R$$

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where T is the matrix $T = (t_{ij})$, $i, j = 1, \dots, n$, in $\text{Mat}(V, \mathbb{C})$. The algebra $A(R)$ is called the algebra of functions on the quantum formal group corresponding to the matrix R . The algebra A is a bialgebra with comultiplication $\Delta: A \rightarrow A \otimes A$ defined by

$$\Delta(1) = 1 \otimes 1 \quad \text{and} \quad \Delta(t_{ij}) = \sum_{k=1}^n t_{ik} \otimes t_{kj}.$$

The counit $\varepsilon: A \rightarrow \mathbb{C}$ is defined by

$$(1.3) \quad \varepsilon(t_{ij}) = \delta_{ij}$$

so that the coassociativity property holds.

Let us now consider the vector representation of the classical Lie algebra $\mathfrak{g} = \mathfrak{sp}(N)$ with $N = 2$. The R -matrix has been calculated [2, 4, 5], and it has the following form:

$$\begin{aligned} R = & q \sum_i E_{ii} \otimes E_{ii} + \sum_{i \neq jj'} E_{jj} \otimes E_{ii} + q^{-1} \sum_{i \neq i'} E_{ii} \otimes E_{i'i'} \\ & + (q - q^{-1}) \sum_{i>j} E_{ij} \otimes E_{ji} - (q - q^{-1}) \sum_{i>j} \varepsilon_i \varepsilon_j q^{\bar{j}-\bar{i}} E_{i'j'} \otimes E_{ij} \end{aligned}$$

where

$$(1.4) \quad \varepsilon_i = \begin{cases} 1, & i = 1, 2, \\ -1, & i = 3, 4, \end{cases} \quad \bar{i} = \begin{cases} i - \frac{1}{2}, & i \leq 2, \\ i + \frac{1}{2}, & i \geq 3, \end{cases} \quad i' = 4 + 1 - i.$$

The corresponding algebra $A(R)$ is the algebra of functions on the q -deformation of the symplectic group $\text{Sp}(2)$.

2. THE QUANTUM $\text{Sp}(2)$

Let $\{e_i\}$, $i = 1, 2, 3, 4$, be the standard basis in $\mathbb{C}^4$ over $\mathbb{C}$. Let q be a real parameter such that $0 < q < 1$. We define the vectors

$$(2.1) \quad \xi_1 = e_1 \otimes e_3 - q e_3 \otimes e_1, \quad \xi_2 = e_2 \otimes e_4 - q e_4 \otimes e_2.$$

Thus the ξ_i belong to the $\mathbb{C}$ -vector space of the alternating 2-tensor $\Lambda^2(\mathbb{C}^4)$ for $i = 1, 2$.

Theorem 1. *Let $y' = (y_q \otimes I)$ be the vector belonging to the $\mathbb{C}$ -vector space of the alternating 4-tensor $\Lambda^4(\mathbb{C}^4) \otimes B(H)$ (see Appendix A1 for the details). Then for any 4×4 matrix u with entries u_i belonging to the bounded linear operators $B(H)$ on a separable Hilbert space H :*

Suppose that the matrix $u = (u_i)$, $i = 1, 2, \dots, 16$, satisfies the following conditions:

$$(2.2) \quad u^* u = I = u u^*$$

where I is the identity matrix on $\text{Mat}(\mathbb{C}^4) \otimes B(H)$, i.e., u is unitary.

The following two conditions are equivalent.

(I)(a) *Quantum symplectic invariance condition.*

$$(2.3) \quad (u \otimes I)(I \otimes u)x = x$$

where $x = \sum_{i=1}^2 \xi_i \otimes \text{id}_{B(H)}$.

(I)(b) *Quantum determinant condition.*

$$(2.4) \quad ((u \otimes I)(I \otimes u) \otimes (u \otimes I)(I \otimes u))y' = y'.$$

(II) *u is of the form*

$$(2.5) \quad u = \begin{pmatrix} u_1 & u_2 & -qu_9^* & -qu_{10}^* \\ u_5 & u_6 & -qu_{13}^* & -qu_{14}^* \\ u_9 & u_{10} & u_1^* & u_2^* \\ u_{13} & u_{14} & u_5^* & u_6^* \end{pmatrix}$$

and satisfies the following equations:

$$\begin{aligned} qu_9u_1 &= u_1u_9, & qu_{13}u_1 &= u_1u_{13}, & qu_{10}u_2 &= u_2u_{10}, \\ qu_{14}u_2 &= u_2u_{14}, & qu_9u_5 &= u_5u_9, & qu_{13}u_5 &= u_5u_{13}, \\ qu_{10}u_6 &= u_6u_{10}, & qu_{14}u_6 &= u_6u_{14}, & qu_1^*u_9 &= u_9u_1^*, \\ qu_1^*u_{10} &= u_{10}u_1^*, & qu_9^*u_2 &= u_2u_9^*, & qu_{10}^*u_2 &= u_2u_{10}^*, \\ qu_5^*u_{13} &= u_{13}u_5^*, & qu_5^*u_{14} &= u_{14}u_5^*, & qu_{13}^*u_6 &= u_6u_{13}^*, \\ qu_{14}^*u_6 &= u_6u_{14}^*, & u_1^*u_{13} &= 0 = u_{13}u_1^*, & u_1^*u_{14} &= 0 = u_{14}u_1^*, \\ u_2^*u_{13} &= 0 = u_{13}u_2^*, & u_2^*u_{14} &= 0 = u_{14}u_2^*, & u_5^*u_9 &= 0 = u_9u_5^*, \\ u_5^*u_{10} &= 0 = u_{10}u_5^*, & u_6^*u_9 &= 0 = u_9u_6^*, & u_6^*u_{10} &= 0 = u_{10}u_6^*, \\ u_9^*u_9 &= u_9u_9^*, & u_9^*u_{10} &= u_{10}u_9^*, & u_9^*u_{13} &= u_{13}u_9^*, \\ u_9^*u_{14} &= u_{14}u_9^*, & u_{10}^*u_{10} &= u_{10}u_{10}^*, & u_{10}^*u_{13} &= u_{13}u_{10}^*, \\ u_{10}^*u_{14} &= u_{14}u_{10}^*, & u_{13}^*u_{13} &= u_{13}u_{13}^*, & u_{13}^*u_{14} &= u_{14}u_{13}^*, \\ u_{14}^*u_{14} &= u_{14}u_{14}^*, & u_2u_1 &= 0 = u_1u_2, & u_5u_1 &= 0 = u_1u_5, \\ u_6u_1 &= u_1u_6, & u_5u_2 &= 0 = u_2u_5, & u_6u_2 &= 0 = u_2u_6, \\ u_6u_5 &= 0 = u_5u_6, & u_{10}u_9 &= 0 = u_9u_{10}, & u_{13}u_9 &= 0 = u_9u_{13}, \\ u_{14}u_9 &= 0 = u_9u_{14}, & u_{13}u_{10} &= 0 = u_{10}u_{13}, & u_{14}u_{10} &= 0 = u_{10}u_{14}, \\ u_{14}u_{13} &= 0 = u_{13}u_{14}, & u_{10}u_{11} &= 0 = u_{11}u_{10}, & u_{14}u_{11} &= 0 = u_{11}u_{14}, \\ u_9u_2 &= 0 = u_2u_9, & u_{13}u_2 &= 0 = u_2u_{13}, & u_{10}u_5 &= 0 = u_5u_{10}, \\ u_{14}u_5 &= 0 = u_5u_{14}, & u_9u_6 &= 0 = u_6u_9, & u_{13}u_6 &= 0 = u_6u_{13}, \end{aligned}$$

$$(2.6) \quad \begin{aligned} [u_1, u_1^*] &= (q - \frac{1}{q})u_9(-qu_9^*), & [u_1, u_2^*] &= (q - \frac{1}{q})u_9(-qu_{10}^*), \\ [u_1, u_5^*] &= (q - \frac{1}{q})u_{13}(-qu_9^*), & [u_1, u_6^*] &= (q - \frac{1}{q})u_{10}(-qu_{13}^*), \\ [u_2, u_2^*] &= (q - \frac{1}{q})u_{10}(-qu_{10}^*), & [u_2, u_5^*] &= (q - \frac{1}{q})u_9(-qu_{14}^*), \\ [u_2, u_6^*] &= (q - \frac{1}{q})u_{14}(-qu_{10}^*), & [u_5, u_5^*] &= (q - \frac{1}{q})u_{13}(-qu_{13}^*), \\ [u_5, u_6^*] &= (q - \frac{1}{q})u_{14}(-qu_{13}^*), & [u_6, u_6^*] &= (q - \frac{1}{q})u_{14}(-qu_{14}^*). \end{aligned}$$

Proof. (I) $\rightarrow$ (II) Let

$$(2.7) \quad u = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 \\ u_5 & u_6 & u_7 & u_8 \\ u_9 & u_{10} & u_{11} & u_{12} \\ u_{13} & u_{14} & u_{15} & u_{16} \end{pmatrix}$$

where $u_i \in B(H)$ with $i = 1, 2, 3, \dots, 16$. We use the fact that u is unitary to simplify (2.3). Multiplying both sides of (2.3) by $u^* \otimes I$, where I denotes the unity of $\mathrm{Mat}(B(H))$, we get

$$(2.8) \quad (I \otimes u)x = (u^* \otimes I)x.$$

We rewrite this equation in matrix form using the basis of $\mathbb{C}^4 \otimes \mathbb{C}^4$ composed of $\{e_i \otimes e_j\}$, $i, j = 1, 2, 3, 4$. The form of the 16×16 matrix on the left-hand side of (2.8) is

$$(2.9) \quad \begin{pmatrix} u & 0_4 & 0_4 & 0_4 \\ 0_4 & u & 0_4 & 0_4 \\ 0_4 & 0_4 & u & 0_4 \\ 0_4 & 0_4 & 0_4 & u \end{pmatrix}$$

where u is given by (2.7) and 0_4 represents the 4×4 matrix with zero entries.

On the right-hand side of (2.8) we obtain instead the following matrix:

$$(2.10) \quad \begin{pmatrix} u_1^* & 0 & 0 & 0 & u_5^* & 0 & 0 & 0 & u_9^* & 0 & 0 & 0 & u_{13}^* & 0 & 0 & 0 \\ 0 & u_1^* & 0 & 0 & 0 & u_5^* & 0 & 0 & 0 & u_9^* & 0 & 0 & 0 & u_{13}^* & 0 & 0 \\ 0 & 0 & u_1^* & 0 & 0 & 0 & u_5^* & 0 & 0 & 0 & u_9^* & 0 & 0 & 0 & u_{13}^* & 0 \\ 0 & 0 & 0 & u_1^* & 0 & 0 & 0 & u_5^* & 0 & 0 & 0 & u_9^* & 0 & 0 & 0 & u_{13}^* \\ u_2^* & 0 & 0 & 0 & 0 & u_6^* & 0 & 0 & 0 & u_{10}^* & 0 & 0 & 0 & u_{14}^* & 0 & 0 \\ 0 & u_2^* & 0 & 0 & 0 & u_6^* & 0 & 0 & 0 & u_{10}^* & 0 & 0 & 0 & u_{14}^* & 0 & 0 \\ 0 & 0 & u_2^* & 0 & 0 & 0 & u_6^* & 0 & 0 & 0 & u_{10}^* & 0 & 0 & 0 & u_{14}^* & 0 \\ 0 & 0 & 0 & u_2^* & 0 & 0 & 0 & u_6^* & 0 & 0 & 0 & u_{10}^* & 0 & 0 & 0 & u_{14}^* \\ u_3^* & 0 & 0 & 0 & 0 & u_7^* & 0 & 0 & 0 & u_{11}^* & 0 & 0 & 0 & u_{15}^* & 0 & 0 \\ 0 & u_3^* & 0 & 0 & 0 & u_7^* & 0 & 0 & 0 & u_{11}^* & 0 & 0 & 0 & u_{15}^* & 0 & 0 \\ 0 & 0 & u_3^* & 0 & 0 & 0 & u_7^* & 0 & 0 & 0 & u_{11}^* & 0 & 0 & 0 & u_{15}^* & 0 \\ 0 & 0 & 0 & u_3^* & 0 & 0 & 0 & u_7^* & 0 & 0 & 0 & u_{11}^* & 0 & 0 & 0 & u_{15}^* \\ u_4^* & 0 & 0 & 0 & 0 & u_8^* & 0 & 0 & 0 & u_{12}^* & 0 & 0 & 0 & u_{16}^* & 0 & 0 \\ 0 & u_4^* & 0 & 0 & 0 & u_8^* & 0 & 0 & 0 & u_{12}^* & 0 & 0 & 0 & u_{16}^* & 0 & 0 \\ 0 & 0 & u_4^* & 0 & 0 & 0 & u_8^* & 0 & 0 & 0 & u_{12}^* & 0 & 0 & 0 & u_{16}^* & 0 \\ 0 & 0 & 0 & u_4^* & 0 & 0 & 0 & u_8^* & 0 & 0 & 0 & u_{12}^* & 0 & 0 & 0 & u_{16}^* \end{pmatrix}$$

The vector x is represented by the column vector

$$x = (0, 0, I, 0, 0, 0, 0, I, -qI, 0, 0, 0, 0, -qI, 0, 0)^T.$$

Therefore, relation (2.3) is satisfied if and only if

$$(2.11) \quad \begin{aligned} -qu_9^* &= u_3, & -qu_{13}^* &= u_7, & -qu_{10}^* &= u_4, & -qu_{14}^* &= u_8, \\ u_{11} &= u_1^*, & u_{15} &= u_5^*, & u_{12} &= u_2^*, & u_{16} &= u_6^*. \end{aligned}$$

This proves (2.5). To conclude this part of the proof we use again the unitarity of u to simplify (2.4)

$$(2.12) \quad (I \otimes I) \otimes (u \otimes I)(I \otimes u)y' = (I \otimes u^*)(u^* \otimes I) \otimes (I \otimes I)y'.$$

From these, by very complicated computations, we get equations (2.6).

(II) $\rightarrow$ (I) Suppose u is of the form (2.5) where $u_1, u_2, u_5, u_6, u_9, u_{10}, u_{13}, u_{14} \in B(H)$ and satisfy (2.6). Equation (2.6) is equivalent to (2.4) where (2.5) holds. $\square$

Theorem 2. Let u be any 4×4 matrix with entries u_i belonging to $B(H)$ where H is a separable Hilbert space. Suppose that u satisfies one of the conditions of Theorem 1. Then $(C^*(u_i), u)$ is a compact quantum group à la Woronowicz, i.e., the symplectic quantum $\text{Sp}(2)$.

Proof. Let u be any 4×4 matrix satisfying Theorem 1 with entries $u_i \in B(H)$ where H is a separable Hilbert space. Then u is of the form (2.5). Define the comultiplication $\Delta: B(H) \rightarrow B(H) \otimes B(H)$ by the formula

$$(2.13) \quad (\text{id} \otimes \Delta)(u) = u_{12}u_{13}.$$

Define the antipode k , $k(u_i) = (u_i^*)^T$, where $*$ means the usual adjoint in $B(H)$. Given u as in (2.5) it is not difficult to check that it is multiplicative. Hence the $*$ -algebra generated by $u_1, u_2, u_5, u_6, u_9, u_{10}, u_{13}, u_{14} \in B(H)$ is a Hopf $*$ -algebra [6]. By using the pentagonal identity

$$(2.14) \quad u_{23}u_{12}u_{23}^* = u_{12}u_{13}$$

and condition (2.3) we have

$$(2.15) \quad u_{12}u_{13}x = x,$$

which implies that $u_{12}u_{23}$ has a fixed point. It follows that

$$(2.16) \quad u_{13}x = u_{12}^*u_{23}u_{12}u_{23}^*x,$$

from which

$$(2.17) \quad u_{13}u_{12}^*x = u_{12}^*u_{23}u_{12}u_{23}^*u_{12}^*x.$$

Finally

$$(2.18) \quad u_{13}u_{12}^*x = u_{12}^*x,$$

so u_{13} has a fixed point. Following the notation of Appendix A2, we have

$$(2.19) \quad u_{13}\theta_{2,\xi}(e_i \otimes u_{ji}^*\xi \otimes \eta) = e_i \otimes u_{ji}^*\xi \otimes \eta.$$

This implies that for every $\eta \in H$

$$(2.20) \quad u(\xi' \otimes \eta) = \xi' \otimes \eta$$

where $\xi' = u_{ji}^*\xi$. Then, by [7, Theorem 4.2], $(C^*(u_i), u)$ is a compact quantum group à la Woronowicz. The reason for which $(C^*(u_i), u)$ is a deformation of a symplectic group will follow from Theorem 1. $\square$

APPENDICES

A1. The twisted alternating vector. By the determinant theorem [8] we have for every $x \in \Lambda^4(\mathbb{C}^4)$, $u: \mathbb{C}^4 \rightarrow \mathbb{C}^4$

$$(u \otimes u \otimes u \otimes u)\xi = (\det u)\xi.$$

In the noncommutative case in which the entries are operators of $B(H)$ we need to find the vector fulfilling the above condition. Let us consider $y' \in \Lambda^4(\mathbb{C}^4) \otimes B(H)$ such that

$$((u \otimes I)(I \otimes u) \otimes (u \otimes I)(I \otimes u))y' = y'.$$

Not every vector will give nontrivial solutions. To find such a vector, let us proceed as follows. Let $\{e_i\}$, $i = 1, 2, 3, 4$, be the standard basis of $\mathbb{C}^4$. Define a twisted alternating vector in $\Lambda^4(\mathbb{C}^4) \otimes B(H)$ by taking

$$y' = (y_q \otimes I)$$

where

$$y_q = \sum_{\sigma \in S_4} (-q)^{l(\sigma)} e_{\sigma 1} \otimes e_{\sigma 2} \otimes e_{\sigma 3} \otimes e_{\sigma 4}.$$

Here $l(\sigma)$ represents the number of inverted pairs while S_4 is the symmetric group of order 4. It is easy to check that y' is an alternating vector. It also gives the twisted unimodularity condition (2.4).

A2. **Notation.** Let H be a Hilbert space. For every $\xi \in H$ we define

$$\theta_\xi, \theta'_\xi \in B(H, H \otimes H)$$

by

$$\theta_\xi(\eta) = \xi \otimes \eta, \quad \theta'_\xi(\eta) = \eta \otimes \xi.$$

We define

$$\theta_{i,\xi} \in B(H \otimes H, H \otimes H \otimes H), \quad i = 1, 2, \text{ or } 3,$$

by

$$\theta_{1,\xi}(\eta \otimes \zeta) = \xi \otimes \eta \otimes \zeta,$$

$$\theta_{2,\xi}(\eta \otimes \zeta) = \eta \otimes \xi \otimes \zeta,$$

$$\theta_{3,\xi}(\eta \otimes \zeta) = \eta \otimes \zeta \otimes \xi.$$

If $V \in B(H \otimes H)$, we define

$$V_{1,2}\theta_{3,\xi} = \theta_{3,\xi}V,$$

$$V_{1,3}\theta_{2,\xi} = \theta_{2,\xi}V,$$

$$V_{2,3}\theta_{1,\xi} = \theta_{1,\xi}V.$$

Let

$$u(e_k \otimes \xi) = \sum_{j=1}^4 e_k \otimes u_{jk}\xi$$

where for convenience the u_{jk} are the generators u_i of the Hopf $*$ -algebra. By $(u_i^*)^T$ we denote the transpose matrix of u in which the entries are the adjoint of the u_i and $\text{Mat}(B(H))$ are the 4×4 matrices with entries belonging to $B(H)$.

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