COMPACT PERTURBATIONS OF FREDHOLM n -TUPLES

RĂZVAN GELCA

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ABSTRACT. Let T be an operator on a Hilbert space. We show that the pair (T, T) can be perturbed to an invertible pair if and only if T is Fredholm of index zero. We also exhibit a large class of Fredholm n -tuples acting on a Banach space which cannot be perturbed by finite rank operators to invertible ones.

INTRODUCTION

It is well known that a Fredholm operator of index zero on a Banach space can be perturbed by a finite rank operator to an invertible one. In [2] it is asked if the same property remains true for commuting pairs of operators, or at least if one can perturb a pair of index zero with *compact* operators to get an invertible one.

There are several properties of the index that are preserved when we pass from one operator to commuting n -tuples of operators, for example, the index is invariant under small-norm perturbations or under perturbations with operators in the norm-closure of finite rank ones; in the Hilbert space case these are exactly the compact operators [1, 4]. It has been proved in [3] that the Koszul complex of a Fredholm n -tuple of index zero has a finite-dimensional perturbation to an exact complex, but the new complex is usually not a Koszul complex of a commuting n -tuple.

In the first part of the present paper we shall prove that on an infinite-dimensional Banach space there exists a large class of Fredholm n -tuples of index zero that cannot be perturbed with finite rank operators to invertible operators. Using the same idea we shall prove in the second part that on an infinite-dimensional Hilbert space no pair of the form $(T, 0)$ with T Fredholm and $\text{ind}(T) \neq 0$ can be perturbed with compact operators to an invertible pair; this will give a negative answer to the question raised by Raúl Curto in [2].

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1. FINITE RANK PERTURBATIONS

In this section we consider the case of (bounded) linear operators acting on an infinite-dimensional Banach space $\mathcal{X}$.

To each commuting n -tuple $T = (T_1, T_2, \dots, T_n)$ of operators on $\mathcal{X}$, we attach a complex of Banach spaces, called the Koszul complex [5], as follows. Let $\Lambda^p = \Lambda^p[e_1, e_2, \dots, e_n]$ be the p -forms on $\mathbb{C}^n$. Define the operator $D_T : \mathcal{X} \otimes \Lambda^p \rightarrow \mathcal{X} \otimes \Lambda^{p+1}$ by $D_T := T_1 \otimes E_1 + T_2 \otimes E_2 + \dots + T_n \otimes E_n$, where $E_i \omega := e_i \omega$, $i = 1, \dots, n$.

The Koszul complex is

$$(1) \quad 0 \rightarrow \mathcal{X} \otimes \Lambda^0 \xrightarrow{D_T} \mathcal{X} \otimes \Lambda^1 \xrightarrow{D_T} \dots \xrightarrow{D_T} \mathcal{X} \otimes \Lambda^n \rightarrow 0.$$

Let $H^p(T)$ be its cohomology spaces. The n -tuple T is called invertible if $H^p(T) = 0$, $0 \leq p \leq n$, and Fredholm if $\dim H^p(T) < \infty$, $0 \leq p \leq n$, in which case we define its index to be $\text{ind } T := \sum_{p=0}^n (-1)^p \dim H^p(T)$.

The Taylor spectrum of T , denoted by $\sigma(T)$, is the set of all $z = (z_1, z_2, \dots, z_n)$ in $\mathbb{C}^n$ such that $z - T = (z_1 - T_1, z_2 - T_2, \dots, z_n - T_n)$ is not invertible. It is known that $\sigma(T)$ is a compact nonvoid set. For any holomorphic map $f : U \rightarrow \mathbb{C}^m$ on a neighborhood U of $\sigma(T)$ one can define $f(T)$ [6]; this functional calculus extends the polynomial calculus. By the spectral mapping theorem [6], $f(\sigma(T)) = \sigma(f(T))$.

If $T = (T_1, T_2, \dots, T_n)$ and $T' = (T_1, T_2, \dots, T_n, S)$ are commuting tuples, then we have a long exact sequence in cohomology

$$(2) \quad \begin{aligned} 0 \rightarrow H^0(T') \rightarrow H^0(T) \xrightarrow{\hat{S}} H^0(T) \rightarrow H^1(T') \rightarrow H^1(T) \\ \rightarrow \dots \rightarrow H^{p-1}(T) \rightarrow H^p(T') \rightarrow H^p(T) \xrightarrow{\hat{S}} H^p(T) \rightarrow \dots \end{aligned}$$

where $\hat{S}$ is the operator induced by $S \otimes 1 : \mathcal{X} \otimes \Lambda^p \rightarrow \mathcal{X} \otimes \Lambda^p$, $0 \leq p \leq n$. If T' is invertible, then $\hat{S}$ is an isomorphism. As a consequence of the long exact sequence, if T is Fredholm, then T' is Fredholm of index zero. We shall prove that there exists a class of $(n+1)$ -tuples of index zero of the form $(T_1, T_2, \dots, T_n, S)$ with $\text{ind}(T_1, T_2, \dots, T_n) \neq 0$ and $(T_1, T_2, \dots, T_n)$ Fredholm that cannot be made invertible by finite rank perturbations.

Lemma 1.1. *Let $(S_1, S_2, \dots, S_n)$ be an invertible commuting n -tuple, and let $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$ be a holomorphic function with $f^{-1}(0) = \emptyset$. Then the m -tuple $f(S_1, S_2, \dots, S_n)$ is invertible.*

Proof. Straightforward from the spectral mapping theorem by noticing the fact that $f^{-1}(0) \cap \sigma(S_1, S_2, \dots, S_n) = \emptyset$.

Lemma 1.2. *Let $(S_1, S_2, \dots, S_n)$ be a Fredholm commuting n -tuple with the property that $\text{ind}(S_1, S_2, \dots, S_n) \neq 0$. Then there exist a sequence of positive integers $\{m_k\}_k$ and $0 \leq p_0 \leq n$ such that $\dim H^{p_0}(S_1^{m_k}, S_2, \dots, S_n) \rightarrow \infty$ for $k \rightarrow \infty$.*

Proof. By Corollary 3.8 in [4] $\text{ind}(S_1^m, S_2, \dots, S_n) = m \cdot \text{ind}(S_1, S_2, \dots, S_n)$, so $\dim H^p(S_1^m, S_2, \dots, S_n)$ cannot all remain bounded.

Theorem 1.1. *Let $(T_1, T_2, \dots, T_n)$ be a Fredholm commuting n -tuple with $\text{ind}(T_1, T_2, \dots, T_n) \neq 0$ and $p \in \mathbb{C}[z_1, z_2, \dots, z_n]$ with $p(0) = 0$. Define*

the operator $T_{n+1} = p(T_1, T_2, \dots, T_n)$. Then there do not exist finite rank operators $R_1, R_2, \dots, R_n, R_{n+1}$ such that $(T_1 + R_1, T_2 + R_2, \dots, T_{n+1} + R_{n+1})$ is an invertible commuting $(n+1)$ -tuple.

Proof. Suppose that such finite rank operators exist, and let $S_i = T_i + R_i$, $1 \leq i \leq n+1$. Applying Lemma 1.1 to the function $f : \mathbf{C}^{n+1} \rightarrow \mathbf{C}^{n+1}$, $f(z_1, z_2, \dots, z_n, z_{n+1}) = (z_1, z_2, \dots, z_n, z_{n+1} - p(z_1, z_2, \dots, z_n))$, we get that $(S_1, S_2, \dots, S_n, R)$ must be invertible, where

$$R = S_{n+1} - p(S_1, S_2, \dots, S_n).$$

Clearly, R is a finite rank operator. By applying Lemma 1.1 to the function $\phi : \mathbf{C}^{n+1} \rightarrow \mathbf{C}^{n+1}$, $\phi(z_1, z_2, \dots, z_n, z_{n+1}) = (z_1^m, z_2, \dots, z_n, z_{n+1})$, we get that $(S_1^m, S_2, \dots, S_n, R)$ is also invertible for every positive integer m .

Let $\{m_k\}_k$ and p_0 be the numbers obtained by applying Lemma 1.2 to the n -tuple $(S_1, S_2, \dots, S_n)$; let $\hat{R} = \hat{R}(m_k, p_0)$ be the operator induced by R on $H^{p_0}(S_1^{m_k}, S_2, \dots, S_n)$. Because $(S_1^{m_k}, S_2, \dots, S_n)$ is invertible, $\hat{R}$ must be an isomorphism for every m_k . But this is impossible since

$$\dim H^{p_0}(S_1^{m_k}, S_2, \dots, S_n) \rightarrow \infty \quad \text{and} \quad \text{rank}(\hat{R}) \leq \binom{n}{p_0} \cdot \text{rank}(R).$$

This proves the theorem.

2. THE MAIN EXAMPLE

In what follows we shall restrict ourselves to bounded linear operators on an infinite-dimensional Hilbert space $\mathcal{H}$. We shall start with a result about the structure of a Fredholm operator of positive index.

Lemma 2.1. *Let T be a Fredholm operator with $\text{ind } T > 0$. Define $\mathcal{H}_n = \ker T^n \ominus \ker T^{n-1}$. Then $\mathcal{H}_n \neq (0)$, $n \geq 2$. Let $T_n = T|_{\ker T^n}$,*

$$(3) \quad T_n : \mathcal{H}_n \oplus \ker T^{n-1} \rightarrow \mathcal{H}_{n-1} \oplus \ker T^{n-2}, \quad T_n = \begin{bmatrix} A_n & 0 \\ B_n & C_n \end{bmatrix}.$$

Then there exists n_0 such that, for $n > n_0$, A_n is an isomorphism.

Proof. Suppose that, for some n , $\mathcal{H}_n = (0)$. Then $\ker T^n = \ker T^{n-1}$. Hence $\ker T^{n+k} = \ker T^n \quad \forall k \geq 0$, but this contradicts the fact that $\lim_{n \rightarrow \infty} \text{ind } T^{n+k} = \infty$.

Since $\mathcal{H}_n \perp \ker T_{n-1}$, $T|_{\mathcal{H}_n}$ is injective and $T\mathcal{H}_n \cap \ker T_{n-2} = (0)$. This shows that A_n is injective. But then the sequence $\{\dim \mathcal{H}_n\}_n$ is decreasing, so it becomes stationary. Let n_0 be such that, for $n > n_0$, $\dim \mathcal{H}_n = \dim \mathcal{H}_{n-1}$. Then, for $n > n_0$, A_n is an injective operator between finite-dimensional spaces of the same dimension, so it is an isomorphism.

Lemma 2.2. *Let T and $\mathcal{H}_n$, $n \geq 2$, be as in the statement of the previous lemma. If S is an operator that commutes with T , then, for all $n \geq 1$, $\ker T^n$ is an invariant subspace for S . Let $S_n = S|_{\ker T^n}$,*

$$(4) \quad S_n : \mathcal{H}_n \oplus \ker T^{n-1} \rightarrow \mathcal{H}_n \oplus \ker T^{n-1}, \quad S_n = \begin{bmatrix} X_n & 0 \\ Y_n & Z_n \end{bmatrix}.$$

Then there is n_0 such that, for $n \geq n_0$, X_n is similar to X_{n_0} .

Proof. The fact that $\ker T^n$ is invariant for S follows from the commutativity. Let A_n and n_0 be as in Lemma 2.1. Then $ST = TS$ implies $S_{n-1}T_n =$

$T_n S_n$, $n \geq 2$. Therefore, $X_{n-1} A_n = A_n X_n$, $n \geq 2$. For $n > n_0$, A_n is an isomorphism; hence, X_n is similar to X_{n-1} . This proves the lemma.

Lemma 2.3. *Let (T, S) be an invertible commuting pair. Then, for any n , $S|_{\ker T^n}$ is an isomorphism for $\ker T^n$.*

Proof. Applying Lemma 1.1 to (T, S) and $f: \mathbb{C}^2 \rightarrow \mathbb{C}^2$, $f(z_1, z_2) = (z_1^n, z_2)$, we get that (T^n, S) is invertible for any n . By the remarks made at the beginning of the first section, $\hat{S}: H^0(T^n) \rightarrow H^0(T^n)$ is an isomorphism. But $H^0(T^n) = \ker T^n$, and the lemma is proved.

Theorem 2.1. *Let T be a Fredholm operator with $\text{ind } T \neq 0$. Then there do not exist compact operators K_1 and K_2 such that $(T + K_1, K_2)$ is an invertible commuting pair.*

Proof. Suppose such K_1 and K_2 exist. Without loss of generality we may assume $\text{ind } T > 0$, otherwise we take T^* instead of T . We can also assume that $K_1 = 0$, otherwise we can denote $T + K_1$ by T and let $K_2 = K$.

Consider the spaces $\mathcal{H}_n$, $n \geq 2$, obtained by applying Lemma 2.1 to T , and let $K_n = K|_{\ker T^n}$. By Lemma 2.2,

$$(5) \quad K_n: \mathcal{H}_n \oplus \ker T^{n-1} \rightarrow \mathcal{H}_n \oplus \ker T^{n-1}, \quad K_n = \begin{bmatrix} X_n & 0 \\ Y_n & Z_n \end{bmatrix}$$

have the property that X_n is similar to X_{n_0} for some n_0 and $n \geq n_0$. Applying Lemma 2.3 we get that the operators X_n , $n \geq 2$, are isomorphisms. If we denote by r the spectral radius of X_{n_0} , then $r > 0$. From the fact that X_n is similar to X_{n_0} for $n \geq n_0$ (so all X_n 's have the same spectral radius), it follows that $\|X_n\| \geq r$.

But $\|K|_{\mathcal{H}_n}\| = \|K_n|_{\mathcal{H}_n}\| \geq \|X_n\| \geq r$ for $n \geq n_0$. Because $\mathcal{H}_n \perp \mathcal{H}_m$, $n \neq m$, and $\mathcal{H}_n \neq (0)$ for any n , it follows that K is not compact, a contradiction. Therefore, such K_1 and K_2 cannot exist.

Corollary 2.1. *The pair (T, T) can be perturbed by compacts to an invertible commuting pair if and only if T can be perturbed by a compact to an invertible operator.*

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DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF IOWA, IOWA CITY, IOWA 52242
E-mail address: rgelca@math.uiowa.edu

INSTITUTE OF MATHEMATICS OF THE ROMANIAN ACADEMY, P.O. Box 1-764, 70700 BUCHAREST, ROMANIA