GLOBAL APPROXIMATION IN HARMONIC SPACES

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ABSTRACT. This paper characterizes, in terms of thinness, compact sets K in a suitable harmonic space Ω which have the following property: functions which are harmonic (resp. continuous and superharmonic) on a neighbourhood of K can be uniformly approximated on K by functions which are harmonic (resp. continuous and superharmonic) on Ω . The corresponding problems of approximating functions which are continuous on K and harmonic (resp. superharmonic) on the interior $\overset{\circ}{K}$ are also solved.

1. INTRODUCTION

The theorems in this paper are inspired by the following celebrated results from the theory of uniform holomorphic approximation. We refer to [Run] and [Con, pp. 198, 201] for Runge's Theorem and to [Mer] or [Rud, Chapter 20] for Mergelyan's Theorem.

Runge's Theorem. *Let Ω be an open subset of $\mathbb{C}$ and K a compact subset of Ω . The following are equivalent:*

- (a) *For every function f which is holomorphic on some neighbourhood of K and every positive number ε , there is a holomorphic function g on Ω such that $|g - f| < \varepsilon$ on K .*
- (b) *There are no components of $\Omega \setminus K$ which are relatively compact in Ω .*

Mergelyan's Theorem. *Let K be a compact subset of $\mathbb{C}$. The following are equivalent:*

- (a) *For every function f which is continuous on K and holomorphic on $\overset{\circ}{K}$ and every positive number ε , there is a complex polynomial g such that $|g - f| < \varepsilon$ on K .*
- (b) *$\mathbb{C} \setminus K$ is connected.*

In this paper we solve the corresponding approximation problems for solutions and continuous supersolutions of a large class of second-order elliptic

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p.d.e.'s; in the case of Laplace's equation, our results are contained in recent work of the first author [Gar]. To be more specific, consider the following differential operator defined on a connected open subset Ω of $\mathbb{R}^n$:

$$L = \sum_i \sum_j a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_j b_j \frac{\partial}{\partial x_j} + c.$$

We assume that $a_{ij} = a_{ji}$ and $c \leq 0$, that $a_{ij} \in C^{2,1}$, that $b_j \in C^{1,1}$, and that $c \in C^{0,1}$, where $C^{k,1}$ denotes the class of functions on Ω which are k times continuously differentiable, the k th-order partial derivatives being locally Lipschitz. Further, we assume that the quadratic form associated with L is positive definite on Ω . The C^2 solutions of the equation $Lu = 0$ will be called *L-harmonic*, and the lower semicontinuous solutions of $Lu \leq 0$ (in the sense of distributions) will be called *L-superharmonic*. We assume that there is a positive *L-superharmonic* function on Ω which is not actually *L-harmonic*.

If $A \subseteq \Omega$, we write $C(A)$ for the collection of continuous real-valued functions on A , $\mathcal{H}(A)$ for the collection of functions which are *L-harmonic* on an open set containing A , and $\mathcal{S}_C(A)$ for the collection of functions which are continuous and *L-superharmonic* on an open set containing A . Let K be a compact subset of Ω . We call (Ω, K) a *Runge pair for L-harmonic* (resp. *continuous L-superharmonic*) *functions* if, for each u in $\mathcal{H}(K)$ (resp. $\mathcal{S}_C(K)$) and each $\varepsilon > 0$, there exists v in $\mathcal{H}(\Omega)$ (resp. $\mathcal{S}_C(\Omega)$) such that $|v - u| < \varepsilon$ on K . Similarly, we call (Ω, K) a *Mergelyan pair for L-harmonic* (resp. *continuous L-superharmonic*) *functions* if, for each u in $\mathcal{E}(K) \cap \mathcal{H}(\overset{\circ}{K})$ (resp. $C(K) \cap \mathcal{S}_C(\overset{\circ}{K})$) and each $\varepsilon > 0$, there exists v in $\mathcal{H}(\Omega)$ (resp. $\mathcal{S}_C(\Omega)$) such that $|v - u| < \varepsilon$ on K . Thus, in each case, we are approximating by globally defined functions. Finally, we define $\hat{K}$ to be the union of K with all components of $\Omega \setminus K$ which are relatively compact in Ω .

In this paper we show that (Ω, K) is a Runge pair for *L-harmonic* functions if and only if $\Omega \setminus \hat{K}$ and $\Omega \setminus K$ are *L-thin* at the same points of K . We also show that (Ω, K) is a Mergelyan pair for *L-harmonic* functions if and only if $\Omega \setminus \hat{K}$ and $\Omega \setminus \overset{\circ}{K}$ are *L-thin* at the same points of K . Further, these assertions remain true if “*L-harmonic*” is replaced by “*continuous L-superharmonic*” in each case. We remark that *L-thinness* is identical to classical thinness (i.e., for Laplace's equation) for many elliptic p.d.e.'s (see [He1, Chapter VII; He2]).

In fact, we shall prove our results in the more general context of BreLOT harmonic spaces, so they also deal with approximation by solutions of p.d.e.'s on Riemannian manifolds (cf. [BG]). The results concerning superharmonic approximation require fewer assumptions, so we deal with them first. For the applicability of our results to the potential theory associated with the operator L described above, see [He1, Chapter VII; Pra].

2. SUPERHARMONIC APPROXIMATION

2.1. Throughout §2, Ω will be a locally compact, locally connected, connected topological space with a countable base for open sets. We suppose that there is a sheaf of continuous functions satisfying Axioms 1, 2, 3 of BreLOT. (For details of BreLOT harmonic spaces, see [Bre, CC, He1].) We assume further that Ω has a base of determining domains (i.e., Axiom D), that the function

1 is superharmonic, and that there is a positive potential on Ω . We define $\hat{K}$, $\mathcal{H}(A)$, $\mathcal{S}_C(A)$, Runge pairs, and Mergelyan pairs as in §1 (without the prefix “ L -”).

Theorem 1. *Let K be a compact subset of Ω . Then (Ω, K) is a Runge pair for continuous superharmonic functions if and only if $\Omega \setminus \hat{K}$ and $\Omega \setminus K$ are thin at the same points of K .*

Theorem 2. *Let K be a compact subset of Ω . Then (Ω, K) is a Mergelyan pair for continuous superharmonic functions if and only if $\Omega \setminus \hat{K}$ and $\Omega \setminus K$ are thin at the same points of K .*

2.2. This section contains some preliminary lemmas. We assume throughout that K is compact and define $W = \hat{K} \setminus K$. A set A in Ω will be called Ω -bounded if $\bar{A}$ is a compact subset of Ω . We note that our hypotheses on Ω imply that this space is metrizable. The proof of the first lemma is reproduced from [Mal, Lemma 1] for the reader's convenience.

Lemma 1. (i) *If ω is an open set which contains K , then there are only finitely many Ω -bounded connected components of $\Omega \setminus K$ which intersect $\Omega \setminus \omega$.*

(ii) *The set $\hat{K}$ is compact.*

To prove this, we first note that $\hat{K}$ is closed, since $\Omega \setminus \hat{K}$ is the union of those (open) connected components of $\Omega \setminus K$ which are not Ω -bounded. In (i) we may assume that ω is Ω -bounded, so $\partial\omega \cap \hat{K}$ is compact. The latter set therefore can be covered by finitely many Ω -bounded components of $\Omega \setminus K$. The remaining (disjoint) Ω -bounded components of $\Omega \setminus K$ must be contained in ω , so (i) is proved. Given any open cover of $\hat{K}$, we can extract a finite subcollection of open sets whose union V contains K . It follows from (i) that $\hat{K} \setminus V$ is compact, so we can obtain a finite subcover of $\hat{K}$, as required.

Lemma 2. *If ω is an open set which contains $\hat{K}$, then there is a compact set C such that $\hat{K} \subset \overset{\circ}{C}$ and $\hat{C} \subset \omega$.*

To prove this, let L be a compact set such that $\hat{K} \subset \overset{\circ}{L}$ and $L \subset \omega$. It follows from Lemma 1 that there are only finitely many Ω -bounded connected components $\delta_1, \dots, \delta_l$ of $\Omega \setminus L$ which intersect $\Omega \setminus \omega$. We fix j in $\{1, \dots, l\}$, choose x_j in $\delta_j \setminus \omega$, and choose y_j to be a point in $\Omega \setminus \hat{L}$ which lies in the connected component U_j of $\Omega \setminus \hat{K}$ which contains x_j . We make an arbitrary choice of metric for Ω and define T_k to be the connected component of $\{x \in \Omega: \text{dist}(x, \hat{K}) > 1/k\}$ which contains x_j . (This exists for all large k .) Then $\bigcup_k T_k = U_j$, so $y_j \in T_k$ for all sufficiently large k . Hence choosing k' sufficiently large and defining

$$C = \{x \in \Omega: \text{dist}(x, \hat{K}) \leq 1/k'\},$$

we obtain a compact set which satisfies $\hat{K} \subset \overset{\circ}{C}$ and $\hat{C} \subset \omega$.

Lemma 3. *If $\Omega \setminus \hat{K}$ and $\Omega \setminus K$ are thin at the same points of K , then W is regular for the Dirichlet problem.*

To establish regularity we must show that, for each x in ∂W , the set $\Omega \setminus W$ is not thin at x . Suppose that $\Omega \setminus W$ is thin at x . Then $\Omega \setminus \hat{K}$ is thin at

x , and, by hypothesis, $\Omega \setminus K$ must also be thin at x . Hence W , which is a subset of $\Omega \setminus K$, is thin at x . This leads to the contradictory conclusion that Ω , being the union of W and $\Omega \setminus W$, is thin at x . Thus the lemma is proved.

Lemma 4. *If $\Omega \setminus \widehat{K}$ and $\Omega \setminus K$ are thin at the same points of K , then $\partial W \subseteq \partial \widehat{K}$.*

To see this, let $A = \partial W \setminus \partial \widehat{K}$. Then A is a relatively open subset of ∂W . Since $\overline{W} \subset \widehat{K}$, we conclude that A is a subset of the interior of $\widehat{K}$. Hence $\Omega \setminus \widehat{K}$ is thin at all points of A . By hypothesis, $\Omega \setminus K$ is thin at all points of A , and the same must be true of W , which is a subset of $\Omega \setminus K$. Axiom D implies that A has harmonic measure zero for each component of W (see [He1, Chapter V]). It follows that each point of A is irregular for the Dirichlet problem on W . Lemma 3 now shows that $A = \emptyset$, as required.

2.3. We turn now to the proof of Theorem 1. Suppose that $\Omega \setminus \widehat{K}$ and $\Omega \setminus K$ are thin at the same points of K , let u be continuous and superharmonic on an open set ω which contains K , and let $\varepsilon > 0$. We denote by $\delta_1, \dots, \delta_l$ the Ω -bounded connected components of $\Omega \setminus K$ which intersect $\Omega \setminus \omega$ (see Lemma 1). (If there are no such components, we proceed immediately to the penultimate paragraph of this section.)

We temporarily fix j in $\{1, \dots, l\}$. There is a continuous positive potential w_j on Ω which has its support S_j contained in δ_j . We define

$$w'_j = w_j - H_{w_j}^{\delta_j} \quad \text{on } \delta_j,$$

where H_w^V denotes the solution to the Dirichlet problem on V with boundary data w . The function w'_j is positive, continuous, and superharmonic on δ_j and continuously vanishes at $\partial \delta_j$ by Lemma 3. Further, w'_j is harmonic on $\delta_j \setminus S_j$. Now let U_j be a δ_j -bounded open set which contains the compact sets S_j and $\delta_j \setminus \omega$. We define

$$a = 1 + \sup\{u(x) : x \in K\} \cup \{0\}$$

and

$$s_j(x) = a - b_j w'_j(x) \quad (x \in \delta_j),$$

where b_j is a positive number chosen sufficiently large to ensure that

$$a - b_j w'_j(x) < u(x) \quad (x \in \partial U_j).$$

Thus if we define

$$u_j(x) = \begin{cases} s_j(x) & (x \in U_j), \\ \min\{s_j(x), u(x)\} & (x \in \delta_j \setminus U_j), \end{cases}$$

we see that u_j is continuous and superharmonic on $\delta_j \setminus S_j$ and that $u_j = u$ at all points of δ_j which are sufficiently close to K .

Having carried out the above constructions for each j in $\{1, \dots, l\}$ we can define

$$\phi(x) = \begin{cases} u_j(x) & (x \in \delta_j; j = 1, \dots, l), \\ u(x) & (x \in \omega \setminus \bigcup_j \delta_j). \end{cases}$$

This function ϕ is continuous and superharmonic on $(\omega \cup \widehat{K}) \setminus \bigcup_j S_j$, and the function $\phi(x) + \sum_{j=1}^l b_j w_j(x)$ is superharmonic on $\omega \cup \widehat{K}$.

We again temporarily fix j in $\{1, \dots, l\}$ and define

$$g_j(x) = w_j(x) - R_{w_j}^{\Omega \setminus \widehat{K}}(x) \quad (x \in \Omega).$$

Then g_j is a nonnegative superharmonic function on δ_j , and its greatest harmonic minorant on δ_j is a bounded harmonic function. However, g_j is also a nonnegative subharmonic (and thus upper semicontinuous) function on $\Omega \setminus S_j$ which vanishes on

$$\{x \in \partial\delta_j: \Omega \setminus \widehat{K} \text{ is not thin at } x\}$$

and hence has limit 0 at all points of this set. If $x \in \partial\delta_j$ and $\Omega \setminus \widehat{K}$ is thin at x , then since $\partial\delta_j \subseteq K$, it follows by hypothesis that $\Omega \setminus K$ is thin at x . Thus δ_j , being a subset of $\Omega \setminus K$, is also thin at x . The set of all such points x must therefore, by Axiom D, form a set of zero harmonic measure for the Dirichlet problem on δ_j . It follows that the greatest harmonic minorant of g_j in δ_j has limit 0 almost everywhere on $\partial\delta_j$ with respect to harmonic measure for δ_j and so is identically 0. Thus g_j is a potential on δ_j , so the greatest harmonic minorant of w_j in δ_j is given by both

$$R_{w_j}^{\Omega \setminus \widehat{K}} \quad \text{and} \quad H_{w_j}^{\delta_j}.$$

Hence

$$g_j = w_j - H_{w_j}^{\delta_j} \quad \text{on } \delta_j,$$

which implies that g_j continuously vanishes on $\partial\delta_j$. We now define

$$c_j = \inf\{g_j(x): x \in S_j\}, \quad A_j = \{x \in \delta_j: g_j(x) \geq \min\{\varepsilon/(2lb_j), c_j/2\}\},$$

and

$$W_p = \{x \in \Omega: \text{dist}(x, \widehat{K}) < 1/p\} \quad (p \geq 1).$$

(We consider only those values of p large enough to ensure that W_p is Ω -bounded.) Then

$$\widehat{R}_{w_j}^{\Omega \setminus W_p} \uparrow R_{w_j}^{\Omega \setminus \widehat{K}} \quad (p \rightarrow \infty).$$

Thus, by Dini's theorem, there exists p' such that the functions h_j defined by

$$h_j(x) = w_j(x) - \widehat{R}_{w_j}^{\Omega \setminus W_{p'}}(x) \quad (x \in \Omega; j = 1, \dots, l)$$

satisfy

$$h_j(x) < g_j(x) + \varepsilon/(2lb_j) \leq \varepsilon/(lb_j) \quad (x \in \partial A_j; j = 1, \dots, l).$$

For each j the function h_j is subharmonic on $\Omega \setminus A_j$ and vanishes outside $W_{p'-1}$. Hence, by the maximum principle,

$$(1) \quad h_j(x) < \varepsilon/(lb_j) \quad (x \in \Omega \setminus A_j; j = 1, \dots, l).$$

We now define

$$(2) \quad v'(x) = \phi(x) + \sum_{j=1}^l b_j h_j(x) \quad (x \in V),$$

where $V = (\omega \cup \widehat{K}) \cap W_{p'}$. This v' is continuous and superharmonic on the open set V (see the conclusion of the third paragraph of this section).

It follows easily from Lemma 2 and [He1, §7] that there is a compact set C which is not thin at any of its boundary points and satisfies $\widehat{K} \subset \overset{\circ}{C}$ and $\widehat{C} \subset V$. Let ω be an Ω -bounded open set such that $\widehat{C} \subset \omega_1$ and $\overline{\omega}_1 \subset V$, and let L be a compact subset of Ω such that $\overline{\omega}_1 \subset \overset{\circ}{L}$. We denote the connected components of $\Omega \setminus \widehat{C}$ by $T_1, \dots, T_k$ (there can only be finitely many of them). For each j in $\{1, \dots, k\}$ let t_j be a continuous positive potential on Ω with compact support in $T_j \setminus L$ and define $t = \sum_j t_j$. Further, we define the continuous function

$$\psi(x) = \alpha - \beta(t(x) - R_t^{\widehat{C}}(x)) \quad (x \in \Omega),$$

where

$$\alpha = 1 + \sup[\{v'(x) : x \in \partial C\} \cup \{0\}]$$

and $\beta > 0$. As in the construction of u_j above, a suitably large choice of β ensures that the function

$$v''(x) = \begin{cases} v'(x) & (x \in \widehat{C}), \\ \min\{v'(x), \psi(x)\} & (x \in \omega_1 \setminus \widehat{C}), \\ \psi(x) & (x \in \Omega \setminus \omega_1) \end{cases}$$

is continuous and superharmonic on a neighbourhood of L and equals v' on K .

Repeating the above argument for an expanding sequence of compact sets L_j in place of L , where $\widehat{L}_j = L_j$ for each j , we obtain a continuous superharmonic function v on Ω which equals v' on K . Further, it follows from (1) and (2) that

$$u = \phi \leq v' = v < \phi + \varepsilon = u + \varepsilon \quad \text{on } K.$$

Thus (Ω, K) is a Runge pair for continuous superharmonic functions, as required.

2.4. The converse assertion in Theorem 1 is an immediate consequence of the two lemmas established below.

Lemma 5. *Let Ω be as in §2.1, let K be a compact subset of Ω , and suppose that (Ω, K) is a Runge pair for continuous superharmonic functions. Then, for each u in $\mathcal{H}(K)$ and each $\varepsilon > 0$, there exists v in $\mathcal{H}(\widehat{K})$ such that $|v - u| < \varepsilon$ on K .*

To prove this, let $u \in \mathcal{H}(K)$ and $\varepsilon > 0$. By hypothesis there exists w_1, w_2 in $\mathcal{S}_C(\Omega)$ such that $u < w_1 < u + \varepsilon$ and $-u < w_2 < -u + \varepsilon$ on K . Thus $-w_2 < w_1$ on K and, hence, by the maximum principle, on $\widehat{K}$. In fact, by continuity, this inequality is valid on an open set V which contains $\widehat{K}$. Since w_1 has a subharmonic minorant on V , it has a greatest harmonic minorant, v say, there. Thus $-w_2 \leq v \leq w_1$ on K , and it follows that $|v - u| < \varepsilon$ on K , as required.

Lemma 6. *Let Ω be as in §2.1 and K a compact subset of Ω . Suppose further that, for each u in $\mathcal{H}(K)$ and each $\varepsilon > 0$, there exists v in $\mathcal{H}(\widehat{K})$ such that $|v - u| < \varepsilon$ on K . Then $\Omega \setminus \widehat{K}$ and $\Omega \setminus K$ are thin at the same points of K .*

To prove Lemma 6, let $\varepsilon > 0$, let $z \in K$, and suppose that $\Omega \setminus K$ is not thin at z . Further, let w be a continuous potential on Ω which characterizes

thinness of sets. (See [He1, Chapter I] for the existence of such a potential.) There is a compact subset C of $\Omega \setminus K$ such that the function $u = R_w^C$ satisfies

$$(3) \quad u(z) > R_w^{\Omega \setminus K}(z) - \varepsilon = w(z) - \varepsilon.$$

Since $u \in \mathcal{H}(K)$, it follows by hypothesis that there exists v in $\mathcal{H}(\widehat{K})$ such that $|v - u| < \varepsilon$ on K . This inequality is actually valid on a neighbourhood of K by continuity. For all sufficiently large m , the set

$$W_m = \{x \in \Omega: \text{dist}(x, \widehat{K}) < 1/m\}$$

is Ω -bounded and $|v - u| < \varepsilon$ on ∂W_m ; whence,

$$|v - \widehat{R}_u^{\Omega \setminus W_m}| = |v - H_u^{W_m}| = |H_{v-u}^{W_m}| < \varepsilon \quad \text{on } W_m.$$

Thus

$$|\widehat{R}_u^{\Omega \setminus W_m} - u| \leq |\widehat{R}_u^{\Omega \setminus W_m} - v| + |v - u| < 2\varepsilon \quad \text{on } K.$$

Letting $m \rightarrow \infty$, it follows that

$$(4) \quad \widehat{R}_u^{\Omega \setminus \widehat{K}} \geq u - 2\varepsilon \quad \text{on } K.$$

Noting that $w \geq R_w^C = u$, we can combine (3) and (4) to obtain

$$\widehat{R}_w^{\Omega \setminus \widehat{K}}(z) \geq \widehat{R}_u^{\Omega \setminus \widehat{K}}(z) > w(z) - 3\varepsilon.$$

Since ε can be arbitrarily small, we conclude that $\widehat{R}_w^{\Omega \setminus \widehat{K}}(z) = w(z)$, so $\Omega \setminus \widehat{K}$ is not thin at z , as required.

2.5. Turning now to the proof of Theorem 2, we record the following particular case of a result of Hansen (see [Han]).

Theorem A. *Let Ω be as in §2.1 and K be a compact subset of Ω . The following are equivalent:*

(a) *For each u in $C(K) \cap \mathcal{S}_C(\overset{\circ}{K})$ and each $\varepsilon > 0$, there exists v in $\mathcal{S}_C(K)$ such that $|v - u| < \varepsilon$ on K .*

(b) *For each u in $C(K) \cap \mathcal{H}(\overset{\circ}{K})$ and each $\varepsilon > 0$, there exists v in $\mathcal{H}(K)$ such that $|v - u| < \varepsilon$ on K .*

(c) *$\Omega \setminus K$ and $\Omega \setminus \overset{\circ}{K}$ are thin at the same points of K .*

Theorem 2 is an immediate consequence of Theorems 1 and A. However, there is a simpler and more direct proof of the “if” part of Theorem 2 which we now indicate.

Let $u \in C(K) \cap \mathcal{S}_C(\overset{\circ}{K})$ and $\varepsilon > 0$. Since

$$\Omega \setminus \widehat{K} \subseteq \Omega \setminus K \subseteq \Omega \setminus \overset{\circ}{K},$$

we observe, by hypothesis, that $\Omega \setminus \widehat{K}$ and $\Omega \setminus K$ are thin at the same points of K . It follows from Lemmas 3 and 4 that, by solving the Dirichlet problem on $\widehat{K} \setminus K$, we can extend u to a function $\tilde{u}$ in $C(\widehat{K}) \cap \mathcal{S}_C(\overset{\circ}{\widehat{K}})$. Also, since

$$\Omega \setminus \widehat{K} \subseteq \Omega \setminus \overset{\circ}{\widehat{K}} \subseteq \Omega \setminus \overset{\circ}{K},$$

we see that $\Omega \setminus \hat{K}$ and $\Omega \setminus \hat{\hat{K}}$ are thin at the same points of K and hence at the same points of $\hat{K}$. Thus, by Theorem A, there exists w in $\mathcal{S}_C(\hat{K})$ such that $|w - \tilde{u}| < \varepsilon$ on $\hat{K}$, whence $|w - u| < \varepsilon$ on K . Arguing as in the last two paragraphs of §2.3, we can construct v in $\mathcal{S}_C(\Omega)$ such that $v = w$ on $\hat{K}$. Hence $|v - u| < \varepsilon$ on K as required.

3. HARMONIC APPROXIMATION

In this section we assume, in addition to the hypotheses of §2.1, that Ω has a base of completely determining domains and that potentials (resp. adjoint potentials) with the same point support are proportional. Further, we assume axiom A^* of quasi-analyticity, i.e., any adjoint harmonic function u on a connected open set w , which vanishes on a neighbourhood of some point of w , must satisfy $u \equiv 0$ on w . These are the hypotheses under which de la Pradelle [Pra] proved his approximation results.

Theorem 3. *Let Ω be as above and K a compact subset of Ω . Then (Ω, K) is a Runge pair for harmonic functions if and only if $\Omega \setminus \hat{K}$ and $\Omega \setminus K$ are thin at the same points of K .*

Theorem 4. *Let Ω be as above and K a compact subset of Ω . Then (Ω, K) is a Mergelyan pair for harmonic functions if and only if $\Omega \setminus \hat{K}$ and $\Omega \setminus \hat{\hat{K}}$ are thin at the same points of K .*

The “if” part of Theorem 3 follows from Theorem 1, Lemmas 5 and 2, and [Pra, Théorème 10]; the converse follows from Lemma 6. Theorem 4 follows from Theorems 3 and A; the “if” part can again be proved more directly, as indicated in §2.5.

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ADDED IN PROOF

Professor Hansen has kindly pointed out that our superharmonic approximation results are implied by theorems in [BH], which were proved in a more abstract setting. However, our main result (Theorem 1) is proved by an elementary approach which seems to be of independent interest.

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