A NOTE ON GENERALIZED SOLUTIONS OF SINGULAR HAMILTONIAN SYSTEMS

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ABSTRACT. We consider T -periodic solutions of singular Hamiltonian systems with weak force

$$\ddot{q} + \nabla V(q, t) = 0,$$

where $V(q, t) \sim -1/|q|^\alpha$ near $q = 0$ with $\alpha \in (0, 2)$. In particular, we study some properties of *generalized* T -periodic solutions, which were introduced by Bahri and Rabinowitz.

0. INTRODUCTION

In this paper we study properties of the generalized solutions of singular Hamiltonian systems which were introduced in [BR]. We consider a Hamiltonian system

$$(HS.1) \quad \ddot{q} + \nabla V(q, t) = 0 \quad \text{in } \mathbf{R},$$

$$(HS.2) \quad q(t + T) = q(t) \quad \text{in } \mathbf{R},$$

where $q = (q_1, q_2, \dots, q_N) \in \mathbf{R}^N$ ($N \geq 3$), $T > 0$, and $V(q, t)$ is a function such that

(V1) $V \in C^2((\mathbf{R}^N \setminus \{0\}) \times \mathbf{R}, \mathbf{R})$ is T -periodic in t ;

(V2) $V(q, t) < 0$ for all (q, t) and $V(q, t), \nabla V(q, t) \rightarrow 0$ as $|q| \rightarrow \infty$ uniformly in t ;

(V3) $V(q, t)$ is of the form

$$V(q, t) = -1/|q|^\alpha + U(q, t),$$

where $\alpha \in (0, 2)$ and $U(q, t) \in C^2((\mathbf{R}^N \setminus \{0\}) \times \mathbf{R}, \mathbf{R})$ is T -periodic in t and satisfies

$$|q|^{\alpha-\rho} U(q, t), |q|^{\alpha-\rho+1} \nabla U(q, t), |q|^{\alpha-\rho+2} \nabla^2 U(q, t), \\ |q|^{\alpha-\rho} U_t(q, t) \rightarrow 0 \quad \text{as } q \rightarrow 0 \text{ uniformly in } t$$

for some $\rho \in (0, \alpha)$.

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In this paper we deal with the case $\alpha \in (0, 2)$, that is, the so-called *weak force* case. The difficulty in proving the existence of periodic solutions of (HS.1)–(HS.2) comes from the lack of the Palais-Smale compactness condition for the corresponding functional

$$I(q) = \int_0^T \left[\frac{1}{2} |\dot{q}|^2 - V(q, t) \right] dt.$$

In [BR] Bahri and Rabinowitz proved the existence of *generalized* T -periodic solutions under weaker conditions

(V1') $V \in C^1((\mathbf{R}^N \setminus \{0\}) \times \mathbf{R}, \mathbf{R})$ is T -periodic in t ;

(V3') $V(q, t) \rightarrow -\infty$ as $q \rightarrow 0$ uniformly in t ;

and (V2). Here a *generalized* T -periodic solution means a function $q(t)$ such that

1° $q(t)$ enters the singularity 0 on a set D of measure 0;

2° $q(t) \in C^2(\mathbf{R} \setminus D, \mathbf{R}^N)$, and $q(t)$ is T -periodic;

3° $q(t)$ satisfies (HS) on $\mathbf{R} \setminus D$;

4° $q(t) \in H^1([0, T], \mathbf{R}^N)$ and $I(q) < \infty$;

5° if $V(q, t)$ is independent of t , then

$$\frac{1}{2} |\dot{q}(t)|^2 + V(q(t)) \equiv \text{const} \quad \text{for } t \in \mathbf{R} \setminus D.$$

They introduced a perturbation of the potential (essentially as in (0.1) below), applied a suitable minimax argument to the perturbed problems, and obtained a generalized T -periodic solution as a limit of classical T -periodic solutions of perturbed problems.

In [T1] we estimated the number of collisions of a generalized solution

$$\nu = \#\{t \in (0, T]; q(t) = 0\}$$

under the conditions (V1)–(V3) (with $\rho = 0$), and we proved

- (i) if $\alpha \in (1, 2)$, then $\nu = 0$; that is, the generalized solution is in fact a classical solution,
- (ii) if $\alpha \in (0, 1]$, then $\nu \leq 1$; that is, the generalized solution has at most one collision in its period.

To obtain the above result, we introduced a perturbation of the potential

$$(0.1) \quad V_\varepsilon(q, t) = V(q, t) - \varepsilon/|q|^2$$

for $\varepsilon \in (0, 1]$. First, we obtain a classical solution $q_\varepsilon(t)$ of

$$(HS.1)_\varepsilon \quad \ddot{q} + \nabla V_\varepsilon(q, t) = 0 \quad \text{in } \mathbf{R},$$

$$(HS.2) \quad q(t+T) = q(t) \quad \text{in } \mathbf{R}.$$

Second, we pass to the limit as $\varepsilon \rightarrow 0$. As to the number of collisions of generalized solutions, we also refer to [B, C, ST, CS1, CS2, T2].

On the other hand, Beaulieu [B] studied the behavior of generalized solutions precisely and proved that if a generalized solution $q(t)$ enters the singularity 0 at time $t = t_0 \in (0, T]$, then the limit

$$(0.2) \quad a_\pm = \lim_{t \rightarrow t_0 \pm 0} \frac{q(t)}{|q(t)|} \in S^{N-1}$$

exists under conditions (V1)–(V3).

The main purpose of this paper is to study the generalized solutions obtained through (HS.1) _{ε} –(HS.2) more precisely; we study the angle between a_+ and a_- . Our main result is

Theorem 0.1. Assume (V1)–(V3). Let $q_\varepsilon(t)$ be a T -periodic solution of (HS.1) _{ε} –(HS.2) that is obtained in [BR]. Assume $q_\varepsilon(t) \rightarrow q(t)$ in L^∞ as $\varepsilon \rightarrow 0$ (if necessary we extract a subsequence) and $q(t)$ enters the singularity 0 at $t = t_0$. Set

$$a_\pm = \lim_{t \rightarrow t_0 \pm 0} \frac{q(t)}{|q(t)|} \in S^{N-1}$$

and

$$\delta_\varepsilon = \min_{t \in (0, T]} |q_\varepsilon(t)| > 0, \quad d = \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{\delta_\varepsilon^{2-\alpha}} \in [0, \infty].$$

(If necessary, we extract a subsequence again so that the above limit exists.) Then

- (i) $d \leq (2 - \alpha)^2 - 1$, and
- (ii) the angle between a_+ and a_- equals $2\pi\sqrt{1+d}/(2-\alpha)$ modulo π .

We will give a proof in §1. In §2 we assume $\alpha = 1$, (V1)–(V3), and (V4) $V(q, t)$ is independent of t and in a neighbourhood of $0 \in \mathbf{R}^N$, $V(q, t)$ is a function of $|q|$, i.e.,

$$V(q, t) = \phi(|q|) \quad \text{for some } \phi(r): (0, \infty) \rightarrow \mathbf{R}$$

in a neighbourhood of $0 \in \mathbf{R}^N$.

We prove that the generalized solution obtained in the above way is a *collision-brake orbit*. More precisely, we prove

Theorem 0.2. Assume (V1)–(V4) and $\alpha = 1$. Let $q_\varepsilon(t)$ be a solution of (HS.1) _{ε} –(HS.2) obtained through a minimax method in [BR]. Suppose $q(t) = \lim_{\varepsilon \rightarrow 0} q_\varepsilon(t)$ enters the singularity 0 at $t = t_0$. Then

$$d = \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{\delta_\varepsilon^{2-\alpha}} = 0 \quad \text{and} \quad a_+ = a_-.$$

Moreover, $q(t)$ is a collision-brake orbit; that is,

$$q(t + t_0) = q(-t + t_0) \quad \text{for all } t.$$

Remark 0.1. Theorem 0.2 is also obtained in [CS1] under a slightly different situation.

For the existence of collision-brake orbits, we refer to [CS2, R].

1. PROOF OF THEOREM 0.1

Let $E = H_T^1(\mathbf{R}, \mathbf{R}^N)$ denote the space of T -periodic functions on $\mathbf{R}$ with values in $\mathbf{R}^N$ under the norm

$$\|q\|_E^2 = \int_0^T |\dot{q}|^2 dt + \int_0^T |q|^2 dt,$$

and let

$$\Lambda = \{q \in E; q(t) \neq 0 \text{ for all } t\}.$$

We define for $\varepsilon \in (0, 1]$

$$I_\varepsilon(q) = \int_0^T \left[\frac{1}{2} |\dot{q}|^2 - V(q, t) + \frac{\varepsilon}{|q|^2} \right] dt \in C^2(\Lambda, \mathbf{R}).$$

Remark 1.1. This functional is slightly different from the functional introduced in [T1], but we can treat it in a similar way (cf. [T2]).

We remark that critical points $q \in \Lambda$ of $I_\varepsilon(q)$ correspond to classical T -periodic solutions of

$$(1.1) \quad \ddot{q} + \nabla V(q, t) + 2\varepsilon q/|q|^4 = 0.$$

Let $q_\varepsilon(t)$ be the critical point of $I_\varepsilon(q)$ obtained in [BR]. In particular, it satisfies

$$(1.2) \quad I_\varepsilon(q_\varepsilon) \in [m, M], \quad I'_\varepsilon(q_\varepsilon) = 0, \quad \text{index } I''_\varepsilon(q_\varepsilon) \leq N - 2,$$

where $m, M > 0$ are constants independent of $\varepsilon \in (0, 1]$ and $\text{index } I''_\varepsilon(q_\varepsilon)$ is the Morse index at q_ε .

As in [T1, T2] we define $\delta_\varepsilon = \min_{t \in (0, T]} |q_\varepsilon(t)| > 0$ and assume $|q_\varepsilon(t)|$ takes its minimum at $t = t_\varepsilon$. We also assume $\delta_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. We introduce a rescaled function

$$(1.3) \quad x_\varepsilon(s) = \delta_\varepsilon^{-1} q_\varepsilon(\delta_\varepsilon^{(\alpha+2)/2} s + t_\varepsilon) \quad \text{for } s \in \mathbf{R}.$$

We extract a subsequence $\varepsilon_n \rightarrow 0$ —we still denote it by ε —such that

$$d = \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{\delta_\varepsilon^{2-\alpha}} \in [0, \infty] \quad \text{exists.}$$

If $d = \infty$, we repeat the argument of the proof of [T2, Proposition 4.1, Case 2]. We can see

$$\lim_{\varepsilon \rightarrow 0} \text{index } I''_\varepsilon(q_\varepsilon) = \infty,$$

and it contradicts (1.2). Therefore, we assume $d < \infty$. Repeating the argument in [T1, T2], we can see

$$(1.4) \quad x_\varepsilon(s) \rightarrow y_{\alpha, d}(s) \quad \text{in } C_{\text{loc}}^2(\mathbf{R}, \mathbf{R}^N),$$

where $y_{\alpha, d}(s)$ is a solution of

$$(1.5) \quad \begin{aligned} \ddot{y} + \alpha y/|y|^{\alpha+2} + 2dy/|y|^4 &= 0 \quad \text{in } t \in \mathbf{R}, \\ y(0) &= e_1, \quad \dot{y}(0) = \sqrt{2(d+1)}e_2, \end{aligned}$$

where $e_1, e_2 \in \mathbf{R}^N$ are suitable vectors satisfying $(e_i, e_j) = \delta_{ij}$.

Clearly $y_{\alpha, d}(s) \in \text{span}\{e_1, e_2\}$ for all $s \in \mathbf{R}$. We use the polar coordinate and write

$$y_{\alpha, d}(s) = r_{\alpha, d}(s)(e_1 \cos \theta_{\alpha, d}(s) + e_2 \sin \theta_{\alpha, d}(s)),$$

where $r_{\alpha, d}(s) > 0$ and $\theta_{\alpha, d}(s) \in \mathbf{R}$ with $\theta_{\alpha, d}(0) = 0$. Modifying the argument in [T1] slightly, we can see

Proposition 1.1. For $\alpha \in (0, 2)$ and $d \in [0, \infty)$,

- (i) $s\dot{r}_{\alpha, d}(s) > 0$ for all $s \neq 0$ and $r_{\alpha, d}(s) \rightarrow \infty$ as $s \rightarrow \pm\infty$;
- (ii) $\dot{\theta}_{\alpha, d}(s) > 0$ for all $s \in \mathbf{R}$;
- (iii) $\theta_\pm(\alpha, d) \equiv \lim_{s \rightarrow \pm\infty} \theta_{\alpha, d}(s) = \pm\pi\sqrt{1+d}/(2-\alpha)$;
- (iv) $\liminf_{\varepsilon \rightarrow 0} \text{index } I''_\varepsilon(q_\varepsilon) \geq (N-2)i(\alpha, d)$, where

$$i(\alpha, d) = \max\{m \in \mathbf{N}; m < 2\sqrt{1+d}/(2-\alpha)\}.$$

Proof. The case $d = 0$ is proved in [T1]. We can get the case $d > 0$ in a similar way. $\square$

Now we can prove the first statement of Theorem 0.1.

Proof of Theorem 0.1(i). Comparing (1.2) and Proposition 1.1(iv), we get $i(\alpha, d) \leq 1$; that is, $2\sqrt{1+d}/(2-\alpha) \leq 2$. Therefore, we obtain $d \leq (2-\alpha)^2 - 1$. $\square$

In what follows, we prove

Proposition 1.2. $a_{\pm} = e_1 \cos \theta_{\pm}(\alpha, d) + e_2 \sin \theta_{\pm}(\alpha, d)$.

We can easily get Theorem 0.1(ii) from the above proposition. In what follows, we assume $t_{\varepsilon} \rightarrow t_0$ as $\varepsilon \rightarrow 0$ and choose a constant $h_0 > 0$ such that

$$q(t) \neq 0 \quad \text{for all } t \in [t_0 - h_0, t_0 + h_0] \setminus \{t_0\}.$$

We study the behavior of $q_{\varepsilon}(t)$ only in the interval $(t_0 - h_0, t_0 + h_0)$. To prove Proposition 1.2, we first observe

Lemma 1.3. (i) *There is a constant $C_0 > 0$ independent of $\varepsilon \in (0, 1]$ such that*

$$(1.6) \quad \left| \frac{1}{2} |\dot{q}_{\varepsilon}(t)|^2 - 1/|q_{\varepsilon}(t)|^{\alpha} + U(q_{\varepsilon}(t), t) - \varepsilon/|q_{\varepsilon}(t)|^2 \right| \leq C_0$$

for all ε and t .

(ii) *There is an $l_0 > 0$ such that $|q_{\varepsilon}(t)| \leq l_0$ implies*

$$\frac{d^2}{dt^2} |q_{\varepsilon}(t)|^2 > 0;$$

that is, if $|q_{\varepsilon}(t)| \leq l_0$, then $|q_{\varepsilon}(t)|$ is increasing (resp. decreasing) for $t > t_{\varepsilon}$ (resp. $t < t_{\varepsilon}$).

Proof. (i) We can get (i) as in [T1, Proposition 1.7].

(ii) From (1.1) and (i) we have

$$\begin{aligned} \frac{1}{2} \frac{d^2}{dt^2} |q_{\varepsilon}|^2 &= (\ddot{q}_{\varepsilon}, q_{\varepsilon}) + |\dot{q}_{\varepsilon}|^2 \\ &\geq \frac{2-\alpha}{|q_{\varepsilon}|^{\alpha}} - (\nabla U(q_{\varepsilon}, t), q_{\varepsilon}) - 2U(q_{\varepsilon}, t) - 2C_0. \end{aligned}$$

Thus by (V3) we can choose l_0 such that

$$(2-\alpha)/|q|^{\alpha} - (\nabla U(q, t), q) - 2U(q, t) - 2C_0 > 0 \quad \text{for } |q| \leq l_0.$$

Thus we get (ii). $\square$

We may assume

$$|q_{\varepsilon}(t)| \leq l_0 \quad \text{for all } t \in [t_0 - h_0, t_0 + h_0]$$

for sufficiently small ε .

Next we define

$$e_{\varepsilon}(t) = (|q_{\varepsilon}|^2 |\dot{q}_{\varepsilon}|^2 - (q_{\varepsilon}, \dot{q}_{\varepsilon})^2)^{1/2} \quad \text{for } t \text{ and } \varepsilon.$$

We remark that $e_{\varepsilon}(t) = |q_{\varepsilon}(t) \times \dot{q}_{\varepsilon}(t)|$ when $N = 3$. Using (1.1), straightforward computation gives us

$$(1.7) \quad |de_{\varepsilon}(t)/dt| \leq |q_{\varepsilon}| |\nabla U(q_{\varepsilon}, t)|.$$

We set

$$\omega_\varepsilon(t) = e_\varepsilon(t)/|q_\varepsilon| |\dot{q}_\varepsilon|.$$

Then $\omega_\varepsilon(t) = \sin \beta$, where β is the angle between $q_\varepsilon(t)$ and $\dot{q}_\varepsilon(t)$. Using (1.1), (1.6), and (1.7), we get

$$\begin{aligned} \frac{d\omega_\varepsilon}{dt}(t) &= \frac{\frac{d}{dt}e_\varepsilon(t)}{|q_\varepsilon| |\dot{q}_\varepsilon|} - \frac{e_\varepsilon(t)}{|q_\varepsilon|^2 |\dot{q}_\varepsilon|^2} \left[\left(\frac{q_\varepsilon}{|q_\varepsilon|}, \dot{q}_\varepsilon \right) |\dot{q}_\varepsilon| + |q_\varepsilon| \left(\frac{\dot{q}_\varepsilon}{|\dot{q}_\varepsilon|}, \ddot{q}_\varepsilon \right) \right] \\ &\leq \frac{2|\nabla U(q_\varepsilon, t)|}{|\dot{q}_\varepsilon|} - \omega_\varepsilon(t) \frac{(q_\varepsilon, \dot{q}_\varepsilon)}{|q_\varepsilon| |\dot{q}_\varepsilon|} \left[|\dot{q}_\varepsilon|^2 - \frac{\alpha}{|q_\varepsilon|^\alpha} - \frac{2\varepsilon}{|q_\varepsilon|^2} \right] \frac{1}{|q_\varepsilon| |\dot{q}_\varepsilon|} \\ &\leq \frac{2|\nabla U(q_\varepsilon, t)|}{|\dot{q}_\varepsilon|} - \omega_\varepsilon(t) \frac{(q_\varepsilon, \dot{q}_\varepsilon)}{|q_\varepsilon| |\dot{q}_\varepsilon|} \left[\frac{2-\alpha}{|q_\varepsilon|^\alpha} - 2U(q_\varepsilon, t) - 2C_0 \right] \frac{1}{|q_\varepsilon| |\dot{q}_\varepsilon|}. \end{aligned}$$

We assume here $(q_\varepsilon(t), \dot{q}_\varepsilon(t)) > 0$. Then we get

$$(1.8) \quad \begin{aligned} \frac{d\omega_\varepsilon}{dt}(t) &\leq \left[-\omega_\varepsilon(t) \sqrt{1 - \omega_\varepsilon(t)^2} \left(\frac{2-\alpha}{|q_\varepsilon|^\alpha} - 2U(q_\varepsilon, t) - 2C_0 \right) \right. \\ &\quad \left. + 2|q_\varepsilon| |\nabla U(q_\varepsilon, t)| \right] \frac{1}{|q_\varepsilon| |\dot{q}_\varepsilon|}. \end{aligned}$$

Now we have

Proposition 1.4. *For any $\eta \in (0, 1)$ there are constants $L_1 > 1$, $l_1 \in (0, l_0)$, and $\varepsilon_1 \in (0, 1)$ such that for $\varepsilon \in (0, \varepsilon_1]$ and $t \in (t_0 - h_0, t_0 + h_0)$*

$$q_\varepsilon(t) \in B_{l_1} \setminus B_{L_1 \delta_\varepsilon} \text{ implies } |\omega_\varepsilon(t)| < \eta,$$

where we use the notation $B_R = \{x \in \mathbf{R}^N; |x| \leq R\}$ for $R > 0$.

Proof. First, choose $l_1 \in (0, l_0)$ such that

$$(1.9) \quad -\eta \sqrt{1 - \eta^2} ((2 - \alpha)/|q|^\alpha - 2U(q, t) - 2C_0) + |q| |\nabla U(q, t)| < 0$$

for all $|q| \leq l_1$ and $t \in (0, T]$.

For the solution $y_{\alpha, d}(s)$ of (1.5), it is clear that

$$\frac{|y_{\alpha, d}(s)|^2 |\dot{y}_{\alpha, d}(s)|^2 - (y_{\alpha, d}(s), \dot{y}_{\alpha, d}(s))^2}{|y_{\alpha, d}(s)|^2 |\dot{y}_{\alpha, d}(s)|^2} \rightarrow 0 \quad \text{as } s \rightarrow \pm\infty$$

and $(y_{\alpha, d}(s), \dot{y}_{\alpha, d}(s)) > 0$ for sufficiently large s . Thus we can choose $s_0 > 0$ such that

$$\left| \frac{|y_{\alpha, d}(s_0)|^2 |\dot{y}_{\alpha, d}(s_0)|^2 - (y_{\alpha, d}(s_0), \dot{y}_{\alpha, d}(s_0))^2}{|y_{\alpha, d}(s_0)|^2 |\dot{y}_{\alpha, d}(s_0)|^2} \right| < \frac{\eta}{2}.$$

We set $L_1 = 2|y_{\alpha, d}(s_0)| > 1$ and $\bar{t}_\varepsilon = t_\varepsilon + \delta_\varepsilon^{(\alpha+2)/2} s_0$. Then

$$\begin{aligned} \omega_\varepsilon(\bar{t}_\varepsilon) &= \frac{|q_\varepsilon(\bar{t}_\varepsilon)|^2 |\dot{q}_\varepsilon(\bar{t}_\varepsilon)|^2 - (q_\varepsilon(\bar{t}_\varepsilon), \dot{q}_\varepsilon(\bar{t}_\varepsilon))^2}{|q_\varepsilon(\bar{t}_\varepsilon)|^2 |\dot{q}_\varepsilon(\bar{t}_\varepsilon)|^2} \\ &= \frac{|x_\varepsilon(s_0)|^2 |\dot{x}_\varepsilon(s_0)|^2 - (x_\varepsilon(s_0), \dot{x}_\varepsilon(s_0))^2}{|x_\varepsilon(s_0)|^2 |\dot{x}_\varepsilon(s_0)|^2} \\ &\rightarrow \frac{|y_{\alpha, d}(s_0)|^2 |\dot{y}_{\alpha, d}(s_0)|^2 - (y_{\alpha, d}(s_0), \dot{y}_{\alpha, d}(s_0))^2}{|y_{\alpha, d}(s_0)|^2 |\dot{y}_{\alpha, d}(s_0)|^2} \in \left(-\frac{\eta}{2}, \frac{\eta}{2} \right) \end{aligned}$$

as $\varepsilon \rightarrow 0$.

So we can assume that $\omega_\varepsilon(\bar{t}_\varepsilon) \leq \frac{\eta}{2}$ for $\varepsilon \in (0, \varepsilon_1]$. We can also assume $(q_\varepsilon(\bar{t}_\varepsilon), \dot{q}_\varepsilon(\bar{t}_\varepsilon)) > 0$. By Lemma 1.3(ii) we remark that $|q_\varepsilon(t)|$ is increasing in $t > \bar{t}_\varepsilon$ as long as $|q_\varepsilon(t)| \leq l_1$.

Arguing indirectly, we assume there is a $t > \bar{t}_\varepsilon$ such that

$$(1.10) \quad q_\varepsilon(t) \in B_{l_1} \quad \text{and} \quad \omega_\varepsilon(t) = \eta.$$

Let $\bar{t} > \bar{t}_\varepsilon$ be the smallest t that satisfies (1.10). By (1.8) and (1.9) we get $d\omega_\varepsilon(\bar{t})/dt < 0$. This contradicts the definition of $\bar{t}$. Therefore, $\omega_\varepsilon(t) < \eta$ for $t \in [\bar{t}_\varepsilon, t_0 + h_0]$ as long as $q_\varepsilon(t) \in B_{l_1}$. Similarly, we can see $\omega_\varepsilon(t) < \eta$ for $t \in [t_0 - h_0, t_\varepsilon - \delta_\varepsilon^{(\alpha+2)/2} s_0]$ as long as $q_\varepsilon(t) \in B_{l_1}$. Since $|q_\varepsilon(\bar{t}_\varepsilon)|/\delta_\varepsilon \rightarrow L_1/2$ as $\varepsilon \rightarrow 0$, we can obtain the desired result. $\square$

We remark here that

$$(1.11) \quad d|q_\varepsilon(t)|/dt = (q_\varepsilon/|q_\varepsilon|, \dot{q}_\varepsilon) \geq \sqrt{1 - \eta^2} |\dot{q}_\varepsilon|$$

provided $\omega_\varepsilon(t) \leq \eta$. From (1.6) we can find $C_1, C'_1 > 0$ such that

$$(1.12) \quad 1/|q_\varepsilon(t)|^{\alpha/2} \leq C_1 |\dot{q}_\varepsilon(t)| \leq C'_1 |d|q_\varepsilon(t)|/dt|.$$

Next we show

Proposition 1.5. *For any $\bar{\eta} > 0$ there are constants $L_2 > 0$, $l_2 \in (0, l_1)$, and $\varepsilon_2 \in (0, \varepsilon_1)$ such that for $\varepsilon \in (0, \varepsilon_2)$*

$$q_\varepsilon(s), q_\varepsilon(t) \in B_{l_2} \setminus B_{L_2 \delta_\varepsilon}$$

and

$$t_\varepsilon < s < t < t_0 + h_0 \quad (\text{or } t_0 - h_0 < s < t < t_\varepsilon)$$

implies

$$|q_\varepsilon(s)/|q_\varepsilon(s)| - q_\varepsilon(t)/|q_\varepsilon(t)|| < \bar{\eta}.$$

Proof. First of all, we have $\omega_\varepsilon(t) < \eta$ if $q_\varepsilon(t) \in B_{l_1} \setminus B_{L_1 \delta_\varepsilon}$. We have

$$\frac{d}{dt} \frac{q_\varepsilon(t)}{|q_\varepsilon(t)|} = \frac{\dot{q}_\varepsilon |q_\varepsilon|^2 - q_\varepsilon (q_\varepsilon, \dot{q}_\varepsilon)}{|q_\varepsilon|^3},$$

i.e.,

$$\left| \frac{d}{dt} \frac{q_\varepsilon(t)}{|q_\varepsilon(t)|} \right| = \frac{e_\varepsilon(t)}{|q_\varepsilon(t)|^2}.$$

Therefore,

$$(1.13) \quad \left| \frac{q_\varepsilon(s)}{|q_\varepsilon(s)|} - \frac{q_\varepsilon(t)}{|q_\varepsilon(t)|} \right| \leq \int_s^t \left| \frac{d}{d\tau} \frac{q_\varepsilon(\tau)}{|q_\varepsilon(\tau)|} \right| d\tau = \int_s^t \frac{e_\varepsilon(\tau)}{|q_\varepsilon(\tau)|^2} d\tau.$$

Now we estimate $e_\varepsilon(\tau)$. We see for $L > L_1$

$$(1.14) \quad \begin{aligned} e_\varepsilon(t_\varepsilon + \delta_\varepsilon^{(\alpha+2)/2} L) / \delta_\varepsilon^{(2-\alpha)/2} &\rightarrow |\dot{y}_{\alpha,d}(L) \times y_{\alpha,d}(L)| \\ &= |\dot{y}_{\alpha,d}(0) \times y_{\alpha,d}(0)| = \sqrt{2(d+1)} \end{aligned}$$

as $\varepsilon \rightarrow 0$.

Since

$$|de_\varepsilon(t)/dt| \leq |q_\varepsilon(t)| |\nabla U(q_\varepsilon, t)| \leq C_2 / |q_\varepsilon|^{\alpha-\rho} \quad \text{for some constant } C_2 > 0,$$

we have

$$\begin{aligned} |e_\varepsilon(\tau)| &\leq |e_\varepsilon(t_\varepsilon + \delta_\varepsilon^{(\alpha+2)/2} L)| + \int_{t_\varepsilon + \delta_\varepsilon^{(\alpha+2)/2} L}^\tau \left| \frac{d}{d\tau} e_\varepsilon(\tau) \right| d\tau \\ &\leq C_3 \delta_\varepsilon^{(2-\alpha)/2} + C_4 \int_{t_\varepsilon + \delta_\varepsilon^{(\alpha+2)/2} L}^\tau \frac{1}{|q_\varepsilon|^{\alpha-\rho}} d\tau. \end{aligned}$$

We use (1.12), (1.14), and the remark that $|q_\varepsilon(t)|$ is increasing to get for $\tau \in [t_\varepsilon + \delta_\varepsilon^{(\alpha+2)/2} L, t_0 + h_0]$ and $|q_\varepsilon(\tau)| \leq l$

$$\begin{aligned} |e_\varepsilon(\tau)| &\leq C_3 \delta_\varepsilon^{(2-\alpha)/2} + C_4' \int_{t_\varepsilon + \delta_\varepsilon^{(\alpha+2)/2} L}^\tau \frac{1}{|q_\varepsilon|^{\alpha/2-\rho}} \frac{d}{d\tau} |q_\varepsilon| d\tau \\ &\leq C_3 \delta_\varepsilon^{(2-\alpha)/2} + C_4'' |q_\varepsilon(\tau)|^{(2-\alpha)/2+\rho}. \end{aligned}$$

Therefore, we get from (1.12) and (1.13) that

$$\begin{aligned} \left| \frac{q_\varepsilon(s)}{|q_\varepsilon(s)|} - \frac{q_\varepsilon(t)}{|q_\varepsilon(t)|} \right| &\leq \int_s^t \frac{C_3 \delta_\varepsilon^{(2-\alpha)/2} + C_4'' |q_\varepsilon|^{(2-\alpha)/2+\rho}}{|q_\varepsilon|^2} d\tau \\ &\leq C_1 \int_s^t \frac{C_3 \delta_\varepsilon^{(2-\alpha)/2} + C_4'' |q_\varepsilon|^{(2-\alpha)/2+\rho}}{|q_\varepsilon|^{(4-\alpha)/2}} \frac{d}{d\tau} |q_\varepsilon(\tau)| d\tau \\ &\leq C_1 \int_{L\delta_\varepsilon}^l \frac{C_3 \delta_\varepsilon^{(2-\alpha)/2} + C_4'' u^{(2-\alpha)/2+\rho}}{u^{(4-\alpha)/2}} du \\ &= C_1 C_3 \int_L^{l/\delta_\varepsilon} \frac{1}{v^{(4-\alpha)/2}} dv + C_1 C_4'' \int_{L\delta_\varepsilon}^l \frac{1}{u^{1-\rho}} du \\ &\leq C_1 C_3 \int_L^\infty \frac{1}{v^{(4-\alpha)/2}} dv + C_1 C_4'' \int_0^l \frac{1}{u^{1-\rho}} du. \end{aligned}$$

We can easily see for sufficiently large $L > 1$ and sufficiently small $l > 0$ that

$$\begin{aligned} \left| \frac{q_\varepsilon(s)}{|q_\varepsilon(s)|} - \frac{q_\varepsilon(t)}{|q_\varepsilon(t)|} \right| &\leq \bar{\eta} \quad \text{if } q_\varepsilon(s), q_\varepsilon(t) \in B_l \\ &\quad \text{and } t_\varepsilon + L\delta_\varepsilon^{(\alpha+2)/2} < s < t < t_0 + h_0. \end{aligned}$$

Similarly we have

$$\begin{aligned} \left| \frac{q_\varepsilon(s)}{|q_\varepsilon(s)|} - \frac{q_\varepsilon(t)}{|q_\varepsilon(t)|} \right| &\leq \bar{\eta} \quad \text{if } q_\varepsilon(s), q_\varepsilon(t) \in B_l \\ &\quad \text{and } t_0 - h_0 < s < t < t_\varepsilon - L\delta_\varepsilon^{(\alpha+2)/2}. \quad \square \end{aligned}$$

End of the proof of Proposition 1.2. Let $b_\pm = e_1 \cos \theta_\pm(\alpha, d) + e_2 \sin \theta_\pm(\alpha, d)$. For any given $\bar{\eta} > 0$ we choose $L_2 > 1$, $l_2 > 0$, and $\varepsilon_2 > 0$ as in Proposition 1.5.

Since

$$y_{\alpha,d}(s)/|y_{\alpha,d}(s)| \rightarrow b_+ \quad \text{as } s \rightarrow \infty,$$

we have for sufficiently large $L > L_2$ and sufficiently small $\varepsilon \in (0, \varepsilon_2)$

$$\left| \frac{q_\varepsilon(t_\varepsilon + \delta_\varepsilon^{(\alpha+2)/2} L)}{|q_\varepsilon(t_\varepsilon + \delta_\varepsilon^{(\alpha+2)/2} L)|} - b_+ \right| < \bar{\eta}.$$

Setting $s = t_\varepsilon + \delta_\varepsilon^{(\alpha+2)/2} L$ in Proposition 1.5, we can see if $q_\varepsilon(t) \in B_{l_2}$ and $t \in (t_\varepsilon + \delta_\varepsilon^{(\alpha+2)/2} L, t_0 + h_0)$, then

$$|q_\varepsilon(t)/|q_\varepsilon(t)| - b_+| < 2\bar{\eta}.$$

Therefore, we can see

$$\{q_\varepsilon(t); t \in (t_\varepsilon, t_0 + h_0)\} \cap B_{l_2} \subset \{x \in B_{l_2}; |x/|x| - b_+| < 2\bar{\eta}\} \cup B_{L\delta_\varepsilon}.$$

Letting $\varepsilon \rightarrow 0$, we have

$$\{q(t); t \in (t_0, t_0 + h_0)\} \cap B_{l_2} \subset \{x \in B_{l_2}; |x/|x| - b_+| \leq 2\bar{\eta}\}.$$

Since $\bar{\eta} > 0$ is arbitrary, we can see

$$\lim_{t \rightarrow t_0+0} \frac{q(t)}{|q(t)|} = b_+.$$

Similarly, we have

$$\lim_{t \rightarrow t_0-0} \frac{q(t)}{|q(t)|} = b_- \quad \square$$

2. PROOF OF THEOREM 0.2

Here, we assume (V1)–(V4) and $\alpha = 1$.

Proof of Theorem 0.2. Suppose $q(t)$ is a generalized solution of (HS.1)–(HS.2) which enters the singularity 0 at $t = t_0$. By (V4), $q(t)$ is a planar motion and $q(t) \times \dot{q}(t)$ is constant near the singularity 0 by the conservation of angular momentum. Since by (1.6)

$$|q(t) \times \dot{q}(t)| \leq |q(t)| |\dot{q}(t)| \leq C|q(t)|^{1-\alpha/2},$$

we can easily see $q(t) \times \dot{q}(t) = 0$ near $t = t_0$; that is, $q(t)$ runs on a line through 0 near $t = t_0$. Clearly such a solution is uniquely determined by the values

$$a = \lim_{t \rightarrow t_0+0} \frac{q(t)}{|q(t)|}, \quad H = \frac{1}{2}|\dot{q}(t)|^2 - \frac{1}{|q(t)|^\alpha} + U(q(t)).$$

Since $\alpha = 1$, we get from Theorem 0.1 $d = 0$ and

$$\lim_{t \rightarrow t_0+0} \frac{q(t)}{|q(t)|} = \lim_{t \rightarrow t_0-0} \frac{q(t)}{|q(t)|}.$$

Therefore, we have

$$q(-t + t_0) = q(t + t_0) \quad \text{for } t \in \mathbf{R}. \quad \square$$

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