SCHUR'S DOUBLE CENTRALIZER THEOREM FOR TRIANGULAR HOPF ALGEBRAS

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ABSTRACT. Let (H, R) be a triangular Hopf algebra and let V be a finite-dimensional representation of H . Following Manin we imitate the standard algebraic constructions in order to define the relativized notions of R -universal enveloping algebras of R -Lie algebras and the R -Lie algebra $\mathrm{gl}_R(V)$. Using Majid's "bosonization" theorem and the above we prove an R -analogue of Schur's double centralizer theorem.

INTRODUCTION

Let V be a finite-dimensional vector space over a field k of characteristic 0. Let m be a positive integer. Then the symmetric group S_m acts on $V^{\otimes m}$ (via the twist map), and the Hopf algebra $U(\mathrm{gl}(V))$ acts on $V^{\otimes m}$ via its comultiplication. In this setting (an equivalent form of) Schur's double centralizer theorem states that S_m and $U(\mathrm{gl}(V))$ are mutual centralizers in $\mathrm{End}_k V^{\otimes m}$. This result was generalized to super vector spaces by [BR], and both these situations were translated into a Hopf algebraic setting and thus given a natural proof by the second author [Fi].

In this paper we deal with a triangular Hopf algebra (H_1, R) over a field k [D] and the category ${}_{H_1}\mathrm{Mod}$ of left H_1 -modules.

In §2 we construct for any R -Lie algebra [Man] the R -analogue of a universal enveloping algebra $U_R(L)$ and show in Theorem 2.6 that $U_R(L)$ is a Hopf algebra in the category ${}_{H_1}\mathrm{Mod}$ (it is not an ordinary Hopf algebra though).

In §3 we consider a finite-dimensional V in the category ${}_{H_1}\mathrm{Mod}$. First we define $\mathrm{gl}_R(V)$ the R -analogue of $\mathrm{gl}(V)$, an R -Lie algebra. Utilizing Majid's recent "bosonization" theorem [Maj2] we form $H = U(\mathrm{gl}_R(V)) \# H_1$, an ordinary Hopf algebra. Next we consider actions of H and kS_m on $V^{\otimes m}$.

On one hand kS_m acts on $V^{\otimes m}$ via the braiding inherited from R ; on the other hand H acts on $V^{\otimes m}$ via the comultiplication of H . In Theorem 3.8 we

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show that in characteristic 0 these representations of kS_m and H are mutual centralizers in $\text{End}_k V^{\otimes m}$, thus giving an R -analogue of Schur's theorem.

1. PRELIMINARIES

Let H be a Hopf algebra over a field k with comultiplication $\Delta: H \rightarrow H \otimes H$, counit $\varepsilon: H \rightarrow k$, and antipode $S: H \rightarrow H$. We use Sweedler's [S] "sigma notation" for Δ ; that is, $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$, for all $h \in H$.

Let $M = {}_H \text{Mod}$ denote the category of left H -modules. For each $V \in M$, $\text{End}_k V \in M$, where for each $f \in \text{End}_k V$ and $h \in H$

$$(1.1) \quad (h \cdot f)(v) \equiv \sum h_{(1)} \cdot (f(S(h_{(2)}) \cdot v)).$$

Moreover, if $V, W \in M$, then $V \otimes W \in M$, where for each $v \otimes w \in V \otimes W$ and $h \in H$

$$(1.2) \quad h \cdot (v \otimes w) = \sum h_{(1)} \cdot v \otimes h_{(2)} \cdot w.$$

The tensor algebra of V , $T(V)$, is thus an H -module algebra. From the properties of the comultiplication, ${}_H \text{Mod}$ is a monoidal category. There does not usually exist, though, a categorical (H -module) isomorphism between $V \otimes W$ and $W \otimes V$ for $V, W \in {}_H \text{Mod}$. The usual twist map τ is not necessarily an H -module map. When the Hopf algebra is (quasi-) triangular such an isomorphism exists and ${}_H \text{Mod}$ becomes a (quasi-) tensor category [Maj1].

Definition 1.3 [D]. A quasi-triangular Hopf algebra is a pair (H, R) , where H is a Hopf algebra and $R = \sum R^{(1)} \otimes R^{(2)} \in H \otimes H$ satisfies the following (with $r = R$):

- (1) R is invertible,
- (2) $(\Delta \otimes \text{id})(R) = \sum R^{(1)} \otimes r^{(1)} \otimes R^{(2)} r^{(2)}$,
- (3) $(\text{id} \otimes \Delta)(R) = \sum R^{(1)} r^{(1)} \otimes r^{(2)} \otimes R^{(2)}$,
- (4) $(\tau \circ \Delta)(h) = R \Delta(h) R^{-1}$ for all $h \in H$.

It is a consequence of the above that

$$R^{-1} = \sum S(R^{(1)}) \otimes R^{(2)} \quad \text{and} \quad \sum \varepsilon(R^{(1)}) R^{(2)} = 1.$$

(H, R) is called triangular if $R^{-1} = \sum R^{(2)} \otimes R^{(1)}$. When (H, R) is quasi-triangular

$$\Psi_{V,W}: V \otimes W \rightarrow W \otimes V \quad \text{given by} \quad v \otimes w \rightarrow \sum R^{(2)} \cdot w \otimes R^{(1)} \cdot v$$

is an H -module isomorphism. Moreover when (H, R) is triangular, it satisfies $\psi_{W,V} \circ \psi_{V,W} = \text{id}$. Sometimes we write ψ_{12} for ψ_{V_1, V_2} .

Note that when H is a cocommutative Hopf algebra, $(H, 1 \otimes 1)$ is triangular and $\psi_{V,W} = \tau$. The rule of thumb in what follows is that whenever one needs to move a v past a w the switching is done via $\psi_{V,W}$. Typically we have

Definition 1.4 [CW, Maj2]. Let A and B be H -module algebras. Then $A \otimes B$ is an H -module algebra by defining the product

$$\mu: A \otimes B \otimes A \otimes B \rightarrow A \otimes B, \quad a \otimes b \otimes c \otimes d \rightarrow \sum a(R^{(2)} \cdot c) \otimes (R^{(1)} \cdot b)c.$$

Definition 1.5. $B \in {}_H \text{Mod}$ is a Hopf algebra in the category if all its structure maps $\Delta_B, \mu_B, S_B, \varepsilon_B$, and u_B (satisfying the usual Hopf algebra diagrams) are H -module maps.

In particular, note that H itself is usually not a Hopf algebra in the category ${}_H \text{Mod}$, although it is an ordinary Hopf algebra.

Majid [Maj2] has recently developed a process by which a Hopf algebra in the category ${}_H \text{Mod}$ is twisted (“bosonized” in his terminology) to become an ordinary Hopf algebra. The “bosonization” theorem was motivated by Fischman’s paper [Fi] which deals with supervector spaces (i.e., spaces constructed by “Bosons” and “Fermions”). Our present paper is a sequel to [Fi], heavily using the bosonization theorem.

We state this theorem in a particular setting (we do not state it in its full generality as it involves various concepts that are out of the scope of this paper).

Theorem 1.6 [Maj2, 4.1]. *Let (H_1, R) be a triangular Hopf algebra over k , and let B be a Hopf algebra in the category ${}_{H_1} \text{Mod}$. If $\psi_{B,B} \circ \Delta_B = \Delta_B$ (i.e., B is “cocommutative” in the category), then the algebra $H = B \# H_1$ is an ordinary triangular Hopf algebra, where (upon identifying B with $B \# 1$ and H_1 with $1 \# H_1$) for all $b \in B$ and $h \in H_1$*

$$\begin{aligned} \hat{\Delta}(b \# h) &= \sum (b_{(1)} \# R^{(2)} h_{(1)}) \otimes (R^{(1)} \cdot b_{(2)} \# h_{(2)}), \quad \text{where } \Delta_B(b) = \sum b_{(1)} \otimes b_{(2)}; \\ S(b \# h) &= \sum S(h_{(2)}) u R^{(1)} \cdot S(b) \# S(R^{(2)} h_{(1)}), \quad \text{where } u = \sum S(R^{(2)}) R^{(1)}; \\ \varepsilon(b \# h) &= \varepsilon(b) \varepsilon(h), \quad u(1) = 1 \# 1, \quad R_{B \# H_1} = R_{H_1}. \end{aligned}$$

The algebra structure on $B \# H_1$ is given as usual by

$$(b \# h)(b' \# h') = \sum b(h_{(1)} \cdot b') \# h_{(2)} h'.$$

2. THE R -UNIVERSAL ENVELOPING ALGEBRA

Let (H_1, R) be a triangular Hopf algebra. In this section we introduce the notion of an R -universal enveloping algebra of an R -Lie algebra. This enveloping algebra, unlike the usual one, is not an ordinary Hopf algebra but will turn out to be a Hopf algebra in the category ${}_{H_1} \text{Mod}$.

Following Manin [Man] we first define an R -Lie algebra to be an object L in ${}_{H_1} \text{Mod}$ together with an H_1 -morphism $[\ , \ ]_R : L \otimes L \rightarrow L$ satisfying

$$(2.1) \quad R\text{-anticommutativity: } [l_1, l_2]_R = - \sum [R^{(2)} \cdot l_2, R^{(1)} \cdot l_1]_R,$$

$$(2.2) \quad R\text{-Jacobi identity:}$$

Let $\{l_1 \otimes l_2 \otimes l_3\}_R$ denote $[l_1, [l_2, l_3]_R]_R$, $S_{312} = \psi_{12} \circ \psi_{23}$, and $S_{231} = \psi_{23} \circ \psi_{12}$ (where $\psi_{23}(l_1 \otimes l_2 \otimes l_3) = \sum (l_1 \otimes (R^{(2)} \cdot l_3) \otimes (R^{(1)} \cdot l_2))$, etc.). Then the following identity holds

$$0 = \{l_1 \otimes l_2 \otimes l_3\}_R + \{S_{312}(l_1 \otimes l_2 \otimes l_3)\}_R + \{S_{231}(l_1 \otimes l_2 \otimes l_3)\}_R, \quad \text{for all } l_i \in L.$$

Note that $[\ , \ ]_R$ being an H_1 -module homomorphism means via (1.2) that $h \cdot [l_1, l_2]_R = \sum [(h_{(1)} \cdot l_1), (h_{(2)} \cdot l_2)]_R$, for all $h \in H$.

Now one can form the derived R -Lie algebra from any H_1 -module algebra A by defining an “inner” R -Lie product [Man]

$$[\ , \ ]_R : A \otimes A \rightarrow A \quad \text{by } [a, b]_R = ab - \sum (R^{(2)} \cdot b)(R^{(1)} \cdot a).$$

This is a morphism in ${}_{H_1} \text{Mod}$ (while the usual commutator is not, unless $R = 1 \otimes 1$). Denote the derived R -Lie algebra by A^- .

Let $(L_i, [\cdot, \cdot]_R^{L_i})$, $i = 1, 2$, be R -Lie algebras. An R -Lie homomorphism is an H_1 -module homomorphism $f: L_1 \rightarrow L_2$ satisfying

$$f([l_1, l_2]_R^{L_1}) = [f(l_1), f(l_2)]_R^{L_2}, \quad \text{for all } l_1, l_2 \in L_1.$$

The following is an R analogue of the classical notion.

Definition 2.3. Let $(L, [\cdot, \cdot]_R)$ be an R -Lie algebra. An R -universal enveloping algebra of L is a pair (U, u) , where U is an associative H_1 -module algebra with 1, $u: L \rightarrow U^-$ is an R -Lie homomorphism, and the following holds: for any associative H_1 -module algebra A and any R -Lie homomorphism $f: L \rightarrow A^-$ there exists a unique algebra homomorphism, which is an H_1 -module homomorphism, $g: U \rightarrow A$, such that the following diagram commutes:

$$\begin{array}{ccc} L & \xrightarrow{u} & U \\ f \downarrow & \searrow g & \\ A & & \end{array}$$

Proposition 2.4 (existence). Let $(L, [\cdot, \cdot]_R)$ be an R -Lie algebra; let $U(L) = T(L)/I$, where I is the ideal of $T(L)$ generated by

$$\left\{ l_1 \otimes l_2 - \sum R^{(2)} \cdot l_2 \otimes R^{(1)} \cdot l_1 - [l_1, l_2]_R, \text{ for all } l_1, l_2 \in L \right\};$$

and let $u: L \rightarrow T(L)/I$ be the canonical map, $l \mapsto l + I = \bar{l}$. Then $(U(L), u)$ is an R -universal enveloping algebra for L .

Proof. First, $U(L)$ is an H_1 -module algebra since $T(L)$ is by (1.2) and I is H_1 -stable, for

$$\begin{aligned} h \cdot (l_1 \otimes l_2 - \sum R^{(2)} \cdot l_2 \otimes R^{(1)} \cdot l_1 - [l_1, l_2]_R) \\ = \sum (h_{(1)} \cdot l_1 \otimes h_{(2)} \cdot l_2 - h_{(1)} R^{(2)} \cdot l_2 \otimes h_{(2)} R^{(1)} \cdot l_1 \\ - [(h_{(1)} \cdot l_1), (h_{(2)} \cdot l_2)]_R) \\ = \sum (h_{(1)} \cdot l_1 \otimes h_{(2)} \cdot l_2 - R^{(2)} h_{(2)} \cdot l_2 \otimes R^{(1)} h_{(1)} \cdot l_1 \\ - [(h_{(1)} \cdot l_1), (h_{(2)} \cdot l_2)]_R) \in I. \end{aligned}$$

The rest is obvious. $\square$

Let us write $[\cdot, \cdot]$ for $[\cdot, \cdot]_R$.

Proposition 2.5. Let L_1 and L_2 be R -Lie algebras. Then there exists an H_1 -module algebra homomorphism

$$g: U(L_1 \oplus L_2) \rightarrow U(L_1) \otimes U(L_2).$$

Proof. Define $f: L_1 \oplus L_2 \rightarrow U(L_1) \otimes U(L_2)$ by $(l, l') \mapsto \bar{l} \otimes 1 + 1 \otimes \bar{l}'$. We claim that f is an R -Lie homomorphism from $L_1 \oplus L_2$ to $(U(L_1) \otimes U(L_2))^-$. Recall that multiplication in $U(L_1) \otimes U(L_2)$ is

$$(a \otimes b)(a' \otimes b') = \sum a(R^{(2)} \cdot a') \otimes (R^{(1)} \cdot b)b',$$

for all $a, a' \in U(L_1)$ and $b, b' \in U(L_2)$. Well,

$$\begin{aligned} f([l, l'], (s, s'))^{L_1 \oplus L_2} &= f([l, s]^{L_1}, [l', s']^{L_2}) \\ &= [\bar{l}, \bar{s}] \otimes 1 + 1 \otimes [\bar{l}', \bar{s}'] = [\bar{l}, \bar{s}] \otimes 1 + 1 \otimes [\bar{l}', \bar{s}']. \end{aligned}$$

Now,

$$\begin{aligned} [f(l, l'), f(s, s')]^{(U(L_1) \otimes U(L_2))^-} &= [(\bar{l} \otimes 1 + 1 \otimes \bar{l}'), (\bar{s} \otimes 1 + 1 \otimes \bar{s}')] \\ &= (\bar{l} \otimes 1 + 1 \otimes \bar{l}')(\bar{s} \otimes 1 + 1 \otimes \bar{s}') \\ &\quad - \sum (R^{(2)} \cdot (\bar{s} \otimes 1 + 1 \otimes \bar{s}'))(R^{(1)} \cdot (\bar{l} \otimes 1 + 1 \otimes \bar{l}')) \\ &= (\bar{l} \otimes 1 + 1 \otimes \bar{l}')(\bar{s} \otimes 1 + 1 \otimes \bar{s}') \\ &\quad - \sum (R^{(2)} \cdot \bar{s} \otimes 1 + 1 \otimes R^{(2)} \cdot \bar{s}')(R^{(1)} \cdot \bar{l} \otimes 1 + 1 \otimes R^{(1)} \cdot \bar{l}') \\ &= \sum \bar{l} \bar{s} \otimes 1 + \bar{l} \otimes \bar{s}' + R^{(2)} \cdot \bar{s} \otimes R^{(1)} \cdot \bar{l}' + 1 \otimes \bar{l}' \bar{s}' \\ &\quad - \sum ((R^{(2)} \cdot \bar{s})(R^{(1)} \cdot \bar{l}) \otimes 1 + R^{(2)} \cdot \bar{s} \otimes R^{(1)} \cdot \bar{l}' \\ &\quad \quad + r^{(2)} R^{(1)} \cdot \bar{l} \otimes r^{(1)} R^{(2)} \cdot \bar{s}' + 1 \otimes (R^{(2)} \cdot \bar{s}')(R^{(1)} \cdot \bar{l}')) \\ &= [\bar{l}, \bar{s}] \otimes 1 + 1 \otimes [\bar{l}', \bar{s}'] \quad [\text{using triangularity of } R]. \end{aligned}$$

Thus f is an R -Lie map.

Now by the universal property of $U(L_1 \oplus L_2)$ there exists an H_1 -module algebra homomorphism $g: U(L_1 \oplus L_2) \rightarrow U(L_1) \otimes U(L_2)$ as claimed. $\square$

Theorem 2.6. *Let (H_1, R) be a triangular Hopf algebra, and let $(L, [\cdot, \cdot]_R)$ be an R -Lie algebra. Let $(U(L), u)$ be the R -universal enveloping algebra of Proposition 2.4. Then $U(L)$ is a Hopf algebra in the category $_{H_1} \text{Mod}$ with*

$$\begin{aligned} \Delta(\bar{l}) &= \bar{l} \otimes 1 + 1 \otimes \bar{l}, \\ S(\bar{l}) &= -\bar{l}, & S(\bar{b}\bar{c}) &= \sum (R^{(2)} \cdot S(\bar{c}))(R^{(1)} \cdot S(\bar{b})), \\ \varepsilon(\bar{l}) &= 0, & \varepsilon(1) &= 1, \end{aligned}$$

for all $l \in L$ and $\bar{b}, \bar{c} \in U(L)$.

Proof. Analogous to the proof in the standard case. Let $d: L \rightarrow L \oplus L$ be the diagonal map $l \rightarrow (l, l)$; d is obviously an R -Lie map. Let f be the map described in Proposition 2.5. Then $f \circ d: L \rightarrow (U(L) \otimes U(L))^-$ is an R -Lie homomorphism; hence, there exists an H_1 -module algebra homomorphism $\Delta: U(L) \rightarrow U(L) \otimes U(L)$ satisfying

$$\Delta(\bar{l}) = ((f \circ d)l) = \bar{l} \otimes 1 + 1 \otimes \bar{l}, \quad \text{for all } l \in L.$$

It is straightforward to check that $(\text{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \text{id}) \circ \Delta$ and $(\varepsilon \otimes \text{id}) \circ \Delta = (\text{id} \otimes \varepsilon) \circ \Delta = \text{id}$. Now consider S which is defined on generators.

$$S(\bar{f}) = -\bar{f}, \quad S(1) = 1,$$

and extended to $U(L)$ via:

$$S(\bar{b}\bar{c}) = \sum (R^{(2)} \cdot S(\bar{c}))(R^{(1)} \cdot S(\bar{b})), \quad \text{for } \bar{b}, \bar{c} \in U(L).$$

It is easy to see that S is a well-defined H_1 -module homomorphism, since if we define $\tilde{S}$ on the free generators of $T(L)$ by $\tilde{S}(l) = -l$, $\tilde{S}(1) = 1$, and set $\tilde{S}(bc) = \sum(R^{(2)} \cdot S(c))(R^{(1)} \cdot S(b))$, then $\tilde{S}$ is an H_1 -module homomorphism which vanishes on I . Thus S is well defined.

To show that S is an antipode, we first note that for generators $\bar{l} \in \bar{L}$ one has

$$\begin{aligned} (\mu(\text{id} \otimes S) \circ \Delta)(\bar{l}) &= (\mu(S \otimes \text{id}) \circ \Delta)(\bar{l}) = 0 = \varepsilon(\bar{l}), \\ (\mu(\text{id} \otimes S) \circ \Delta)(1) &= (\mu(S \otimes \text{id}) \circ \Delta)(1) = 1 = \varepsilon(1), \end{aligned}$$

which is a straightforward computation, and then we use the following

Claim. If H is a bialgebra in the category $_{H_1} \text{Mod}$ and if

$$(\mu(\text{id} \otimes S) \circ \Delta)(a) = \varepsilon(a), \quad (\mu(\text{id} \otimes S) \circ \Delta)(b) = \varepsilon(b),$$

and $S(ab) = \sum(R^{(2)} \cdot S(b))(R^{(1)} \cdot S(a))$, then $(\mu(\text{id} \otimes S) \circ \Delta)(ab) = \varepsilon(ab)$.

Proof of Claim.

$$\begin{aligned} &(\mu(\text{id} \otimes S) \circ \Delta)(ab) \\ &= \mu(\text{id} \otimes S) \sum (a_{(1)}(R^{(2)} \cdot b_{(1)}) \otimes (R^{(1)} \cdot a_{(2)})b_{(2)}) \\ &= \mu \sum (a_{(1)}(R^{(2)} \cdot b_{(1)}) \otimes r^{(2)} \cdot S(b_{(2)})(r^{(1)}R^{(1)} \cdot S(a_{(2)}))) \\ &= \sum a_{(1)}(R^{(2)} \cdot b_{(1)})(R^{(2)} \cdot S(b_{(2)}))(R^{(1)} \cdot S(a_{(2)})) \quad (\text{using property (2) of } R) \\ &= \sum a_{(1)}(R^{(2)} \cdot (b_{(1)}S(b_{(2)})))(R^{(1)} \cdot S(a_{(2)})) \\ &= \sum a_{(1)}(R^{(2)} \cdot \varepsilon(b))(R^{(1)} \cdot S(a_{(2)})) \quad (\text{by assumption}) \\ &= \varepsilon(a)\varepsilon(b) \quad (\text{since } \sum \varepsilon(R^{(2)})R^{(1)} = 1). \end{aligned}$$

Using the claim we deduce that S is an antipode on $U(L)$.

Remark 2.7. If A is an H -module algebra and A^- its derived Lie algebra, then the map $u: A^- \rightarrow U(A^-)$ is an injection.

Proof. The identity map from A^- to A^- is a Lie map; hence by the universality of $U(R)$ there is an algebra map $p: U(A^-) \rightarrow A$ such that $p \circ u = \text{id}$. Thus u is 1-1.

3. AN R -ANALOGUE OF SCHUR'S DOUBLER CENTRALIZER THEOREM

Let us first recall Schur's double centralizer theorem in its classical form.

Let V be a finite-dimensional vector space over a field k of characteristic 0. Let m be a positive integer, and define a representation θ of the symmetric group S_m on $V^{\otimes m}$ by $\sigma \cdot (v_1 \otimes \cdots \otimes v_m) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}$. In particular, $(i, i+1) \cdot (v_1 \otimes \cdots \otimes v_m) = v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_m$; that is, this representation is essentially defined by the twist map τ . It extends to an action of the Hopf algebra kS_m on $V^{\otimes m}$ and hence on $\text{End}_k V^{\otimes m}$ (as in (1.1)). On the other hand consider the Hopf algebra $U(\text{gl}(V))$, the universal enveloping algebra of the Lie algebra $\text{gl}(V)$. It acts on V by composition, and since it is a Hopf algebra, it acts on $V^{\otimes m}$ by (1.2). We denote this representation by ρ^m . It entails using $\Delta^{m-1}(h)$, for all $h \in U(\text{gl}(V))$.

Theorem 3.1 (Schur's double centralizer theorem in equivalent form). *In the above notation we have*

$$\text{End}_{\theta(kS_m)} V^{\otimes m} = \rho^m(U(\text{gl}(V))) \quad \text{and} \quad \text{End}_{\rho^m(U(\text{gl}(V)))} V^{\otimes m} = \theta(kS_m).$$

Berele and Regev [BR] have extended this to super vector spaces, and [Fi] proved their result in a Hopf algebraic setting. In this paper we prove a double centralizer theorem for finite-dimensional $V \in {}_{H_1}\text{Mod}$, where (H_1, R) is a triangular Hopf algebra. The previous results will follow immediately from this theorem. Let us apply the above.

Let $V \in {}_{H_1}\text{Mod}$ be a finite-dimensional vector space over k . By (1.1) $\text{End}_k V \in {}_{H_1}\text{Mod}$. Let $\text{gl}_R(V) \equiv (\text{End}_k V)^-$, and let $U(\text{gl}_R(V))$ be the R -universal enveloping algebra described in Theorem 2.6.

Remark 3.2. In particular, if $H_1 = k$, then $U(\text{gl}_R(V))$ is just $U(\text{gl}(V))$. As for the super case, following Majid's construction [Maj2], let $H_1 = k\mathbb{Z}_2$ ($\text{char } k \neq 2$) be the group algebra of the group $\mathbb{Z}_2 = \{1, g\}$ with the nonstandard triangular structure

$$R = \frac{1}{2}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g).$$

The category of $k\mathbb{Z}_2$ -modules is precisely the category of super vector spaces, where $g \cdot v = (-1)^{|v|}v$ (where $|v| = \text{degree } v$).

If $V \in {}_{k\mathbb{Z}_2}\text{Mod}$ is finite dimensional, then $U(\text{gl}_R(V)) = U(\text{pl}(V))$, where $\text{pl}(V)$ is the Lie superalgebra of $\mathbb{Z}_2$ -graded endomorphisms of V .

Combining Theorem 2.6 and the bosonization theorem we have

Corollary 3.3. *Let (H_1, R) and $U(\text{gl}_R(V))$ be as in Theorem 2.6 with $\text{gl}_R(V) = (\text{End}_k V)^-$, V a finite-dimensional vector space over k . Then*

$$H = U(\text{gl}_R(V)) \# H_1$$

is an ordinary triangular Hopf algebra with

$$\Delta(f\#1) = (f\#1) \otimes (1\#1) + \sum (1\#R^{(2)}) \otimes (R^{(1)} \cdot f\#1), \quad \Delta(1\#h) = \Delta_{H_1}(h),$$

$$S(f\#1) = - \sum y R^{(1)} \cdot f\#S(R^{(2)}) \quad (\text{where } y = \sum S(R^{(2)})R^{(1)}),$$

$$S(1\#h) = 1\#S(h), \quad \varepsilon(f\#1) = 0, \quad \varepsilon(1\#h) = \varepsilon(h), \quad R_H = R_{H_1},$$

for all $f \in \text{gl}_R(V)$, $h \in H_1$.

Since $u: \text{gl}_R(V) \rightarrow U(\text{gl}_R(V))$ is an injection (see Remark 2.7) we write f for $\bar{f}$. Define an obvious representation of H on V : $\rho: H \rightarrow \text{End}_k(V)$ given by

$$\rho(f_1 \cdots f_i \# 1) = f_1 \circ \cdots \circ f_i, \quad \text{for all } f_i \in \text{End}_k V$$

and

$$(\rho(1\#h))(v) = h \cdot v, \quad \text{for all } h \in H_1, v \in V.$$

Extend ρ to H by $\rho(b\#h) = \rho(b) \circ \rho(h)$, for all $b = f_1 \cdots f_i \in U(\text{gl}_R(V))$, $h \in H_1$. To show that ρ is a representation one has to show that $\rho((1\#h)(b\#1)) = \rho(h) \circ \rho(b)$. Indeed,

$$(1\#h)(b\#1) = \sum h_{(1)} \cdot b\#h_{(2)}.$$

Hence

$$\begin{aligned}
 (\rho(1\#h)(b\#1))(v) &= \sum (\rho(h_{(1)} \cdot b) \rho(h_{(2)}))(v) \\
 &= \sum (h_{(1)} \cdot (f_1 \circ \cdots \circ f_i))(h_{(2)} \cdot v) \\
 &= \sum h_{(1)} \cdot (f_1 \cdots f_i(S h_{(2)} h_{(3)} \cdot v)) \\
 &= h \cdot (f_1 \cdots f_i(v)) = (\rho(h) \circ \rho(b))(v).
 \end{aligned}$$

Thus ρ is a representation of $U(\mathfrak{gl}_R(V))\#H_1$ on V . From now on, denote $\rho(g)$ by $\underline{g}$ for all $g \in U(\mathfrak{gl}_R(V))\#H_1$; and since ρ is onto we will denote by $\underline{f}$ all elements of $\text{End}_k V$. The representation ρ induces a representation ρ^m on $V^{\otimes m}$ as in (1.2); i.e.,

$$\begin{aligned}
 (3.4) \quad \rho^m(g)(v_1 \otimes \cdots \otimes v_m) &= \underline{g_{(1)}}(v_1) \otimes \cdots \otimes \underline{g_{(m)}}(v_m), \\
 \text{where } \Delta^{m-1}(g) &= \sum g_{(1)} \otimes \cdots \otimes g_{(m)}.
 \end{aligned}$$

Remark. Some caution is necessary here, for if $x = b\#h \in H$, then one must distinguish between x and $\underline{x}\#1$; $\Delta(x)$ could very well be different from $\Delta(\underline{x}\#1)$, and so $\rho^m(x) \neq \rho^m(\underline{x}\#1)$. Even in the trivial example if $x = 1\#1$, then $\underline{x}\#1 = \text{id}\#1$ and thus

$$\Delta(\underline{x}\#1) = (\text{id}\#1) \otimes (1\#1) + (1\#1) \otimes (\text{id}\#1).$$

So $\rho^2(\underline{x}\#1)(v_1 \otimes v_2) = 2(v_1 \otimes v_2)$ while $\Delta(x) = (1\#1) \otimes (1\#1)$, and so

$$\rho^2(x)(v_1 \otimes v_2) = v_1 \otimes v_2$$

(this of course is well known in the representation theory of Lie algebras).

Let us now consider the symmetric group S_m . There is a representation of S_m on $V^{\otimes m}$ induced by $\psi_{V,V}$ (rather than τ). Specifically, define $\phi: kS_m \rightarrow \text{End } V^{\otimes m}$ by

$$(3.5) \quad (i, i+1) \cdot (v_1 \otimes \cdots \otimes v_m) = \sum v_1 \otimes \cdots \otimes R^{(2)} \cdot v_{i+1} \otimes R^{(1)} \cdot v_i \otimes \cdots \otimes v_m;$$

extend this multiplicatively to S_m and linearly to kS_m .

The action of the Hopf algebra kS_m on $V^{\otimes m}$ now induces an action of kS_m on $\text{End}_k V^{\otimes m}$ as in (1.1). That is, for all $\underline{f}_j \in \text{End}_k V$ and $v_j \in V$ let

$$\begin{aligned}
 ((i, i+1) \cdot (\underline{f}_i \otimes \cdots \otimes \underline{f}_m))(v_1 \otimes \cdots \otimes v_m) \\
 &= (i, i+1) \cdot ((\underline{f}_1 \otimes \cdots \otimes \underline{f}_m)((i, i+1) \cdot (v_1 \otimes \cdots \otimes v_m))) \\
 &= (i, i+1) \cdot \left((\underline{f}_1 \otimes \cdots \otimes \underline{f}_m) \sum (v_1 \otimes \cdots \otimes R^{(2)} \cdot v_{i+1} \otimes R^{(1)} \cdot v_i \otimes \cdots \otimes v_m) \right) \\
 &= \sum \underline{f}_1(v_1) \otimes \cdots \otimes r^{(2)} \cdot (\underline{f}_{i+1}(R^{(1)} \cdot v_i)) \otimes r^{(1)} \cdot (\underline{f}_i(R^{(2)} \cdot v_{i+1})) \otimes \cdots \otimes \underline{f}_m(v_m) \\
 &= \sum (\underline{f}_1 \otimes \cdots \otimes \underline{r}^{(2)} \underline{f}_{i+1} \underline{R}^{(1)} \otimes \underline{r}^{(1)} \underline{f}_i \underline{R}^{(2)} \otimes \cdots \otimes \underline{f}_m)(v_1 \otimes \cdots \otimes v_m)
 \end{aligned}$$

(with $r = R$). Thus we have shown:

$$(3.6) \quad (i, i+1) \cdot (\underline{f}_1 \otimes \cdots \otimes \underline{f}_m) = \sum \underline{f}_1 \otimes \cdots \otimes \underline{r}^{(2)} \underline{f}_{i+1} \underline{R}^{(1)} \otimes \underline{r}^{(1)} \underline{f}_i \underline{R}^{(2)} \otimes \underline{f}_{i+1} \otimes \cdots \otimes \underline{f}_m.$$

In the following lemma we have repeated occurrences of R which we shall denote by $R_1, \dots, R_j$, where $R = R_i$, for all i . As a matter of convenience we shall write $R_0^{(2)}$ for empty word and $R_0^{(1)}$ for 1.

Lemma 3.7. *Let $f \in \mathfrak{gl}_R(V)$, and let $\Delta = \Delta_H$, where $H = U(\mathfrak{gl}_R(V))\#H_1$. Then*

- (1) $\Delta^m(f\#1) = \sum_{j=0}^m (1\#R_1^{(2)}) \otimes \cdots \otimes (1\#R_j^{(2)}) \otimes (R_j^{(1)} \cdots R_1^{(1)} \cdot f\#1) \otimes 1^{\otimes m-j}$,
- (2) $\rho^{m+1}(f\#1) = (1 + (2, 1) + \cdots + (m+1, m) \cdots (2, 1)) \cdot (\underline{f} \otimes \text{id}^{\otimes m})$.

Proof. First we claim that

$$(*) \quad \begin{aligned} & (i, i+1) \cdot \underline{f}_1 \otimes \cdots \otimes \underline{f}_i \otimes \text{id} \otimes \cdots \otimes \underline{f}_m \\ &= \sum \underline{f}_1 \otimes \cdots \otimes \underline{f}_{i-1} \otimes R^{(2)} \otimes R^{(1)} \cdot \underline{f}_i \otimes \underline{f}_{i+1} \otimes \cdots \otimes \underline{f}_m. \end{aligned}$$

Well, by (3.5)

$$\begin{aligned} & (i, i+1) \cdot (\underline{f}_1 \otimes \cdots \otimes \underline{f}_i \otimes \text{id} \otimes \cdots \otimes \underline{f}_m) \\ &= \sum \underline{f}_1 \otimes \cdots \otimes \underline{f}_i \otimes \underline{f}_i R^{(1)} \otimes R^{(1)} \underline{f}_i R^{(2)} \otimes \underline{f}_{i+1} \otimes \cdots \otimes \underline{f}_m, \end{aligned}$$

so $(*)$ will follow if we show that $\sum \underline{f}_i R^{(1)} \otimes R^{(1)} \underline{f}_i R^{(2)} = \sum \underline{f}_i R^{(2)} \otimes R^{(1)} \cdot \underline{f}_i$, for all $\underline{f} \in \text{End}_k V$. Now $\sum (\underline{f}_i R^{(2)} \otimes R^{(1)} \cdot \underline{f}_i)(v_1 \otimes v_2) = \sum R^{(2)} \cdot v_1 \otimes R^{(1)} \cdot (\underline{f}(S R^{(1)} \cdot v_2)) = \sum R^{(2)} r^{(2)} \cdot v_1 \otimes R^{(1)} \cdot (\underline{f}(S r^{(1)} \cdot v_2))$ (by property (2) of R). But since (H, R) is triangular, $\sum S(r^{(1)}) \otimes r^{(2)} = R^{-1} = R^r = \sum r^{(2)} \otimes r^{(1)}$. Substituting this, the above equals

$$\sum R^{(2)} r^{(1)} \cdot v_1 \otimes R^{(1)} \cdot (\underline{f}(r^{(2)} \cdot v_2)) = \sum (\underline{f}_i R^{(2)} r^{(1)} \otimes R^{(1)} \underline{f}_i R^{(2)})(v_1 \otimes v_2),$$

as claimed.

Now we prove the lemma by induction on m .

For $m = 1$, $\Delta(f\#1) = (f\#1) \otimes 1 + \sum (1\#R^{(2)}) \otimes (R^{(1)} \cdot f\#1)$. Hence

$$\begin{aligned} \rho^2(\underline{f}\#1) &= \underline{f} \otimes \text{id} + \sum \underline{f}_i R^{(2)} \otimes R^{(1)} \cdot \underline{f}_i \\ &= 1(\underline{f} \otimes \text{id}) + (2, 1) \cdot (\underline{f} \otimes \text{id}) \quad (\text{by } (*)) \\ &= (1 + (2, 1)) \cdot (\underline{f} \otimes \text{id}) \end{aligned}$$

Assume the lemma is true for Δ^{m-1} and ρ^m . Then

$$\begin{aligned} \Delta^m(f\#1) &= (\text{id}^{\otimes m-1} \otimes \Delta)(\Delta^{m-1}(f)) \\ &= (\text{id}^{\otimes m-1} \otimes \Delta) \sum_{j=0}^{m-1} (1\#R_1^{(2)}) \otimes \cdots \otimes (1\#R_j^{(2)}) \otimes (R_j^{(1)} \cdots R_1^{(1)} \cdot f\#1) \otimes 1^{\otimes m-1-j} \\ &= (\text{id}^{\otimes m-1} \otimes \Delta) \left(\sum_{j=0}^{m-2} (1\#R_1^{(2)}) \otimes \cdots \otimes (1\#R_j^{(2)}) \otimes (R_j^{(1)} \cdots R_1^{(1)} \cdot f\#1) \otimes 1^{\otimes m-1-j} \right) \\ &\quad + (1\#R_1^{(2)}) \otimes \cdots \otimes (1\#R_{m-1}^{(2)}) \otimes (R_{m-1}^{(1)} \cdots R_1^{(1)} \cdot f\#1) \\ &= \sum_{j=0}^{m-2} (1\#R_1^{(2)}) \otimes \cdots \otimes (1\#R_j^{(2)}) \otimes (R_j^{(1)} \cdots R_1^{(1)} \cdot f\#1) \otimes 1^{\otimes m-j-1} \\ &\quad + (1\#R_1^{(2)}) \otimes \cdots \otimes (1\#R_{m-1}^{(2)}) \otimes \Delta(R_{m-1}^{(1)} \cdots R_1^{(1)} \cdot f\#1). \end{aligned}$$

But

$$\begin{aligned} \Delta(R_{m-1}^{(1)} \cdots R_1^{(1)} \cdot f\#1) &= ((R_{m-1}^{(1)} \cdots R_1^{(1)} \cdot f\#1) \otimes 1) \\ &\quad + (1\#R_m^{(2)}) \otimes (R_m^{(1)} R_{m-1}^{(1)} \cdots R_1^{(1)} \cdot f\#1). \end{aligned}$$

Hence, the whole expression equals

$$\sum_{j=0}^m (1 \# R_1^{(2)}) \otimes \cdots \otimes (1 \# R_j^{(2)}) \otimes (R_j^{(1)} \cdots R_1^{(1)} \cdot f \# 1) \otimes 1^{\otimes m-j}.$$

We have proved (1). Now, for (2) we show by induction that

$$(**) \quad (m+1, m) \cdots (2, 1) \cdot (\underline{f} \otimes \underline{\text{id}}^{\otimes m}) = \sum \underline{R}_1^{(2)} \otimes \cdots \otimes \underline{R}_m^{(2)} \otimes R_m^{(1)} \cdots R_1^{(1)} \cdot \underline{f}.$$

This is true for $m = 1$; assume for m . Then

$$\begin{aligned} & (m+1, m) \cdots (2, 1) (\underline{f} \otimes \underline{\text{id}}^{\otimes m}) \\ &= (m+1, m) \cdot ((m, m-1) \cdots (2, 1) \cdot (\underline{f} \otimes \underline{\text{id}}^{\otimes m-1} \otimes \underline{\text{id}})) \\ &= (m+1, m) \cdot \sum (\underline{R}_1^{(2)} \otimes \cdots \otimes \underline{R}_{m-1}^{(2)} \otimes R_{m-1}^{(1)} \cdots R_1^{(1)} \cdot \underline{f} \otimes \underline{\text{id}}) \\ & \quad \text{(by the induction hypothesis)} \\ &= \sum \underline{R}_1^{(2)} \otimes \cdots \otimes \underline{R}_{m-1}^{(2)} \otimes \underline{R}_m^{(2)} \otimes R_m^{(1)} \cdots R_1^{(1)} \cdot \underline{f} \quad \text{(by *)}, \end{aligned}$$

as claimed. To show (2),

$$\begin{aligned} & [1 + (2, 1) + \cdots + (m, m-1) \cdots (2, 1) + (m+1, m) \cdots (2, 1)] \cdot (\underline{f} \otimes \underline{\text{id}}^{\otimes m}) \\ &= \rho^m(f \# 1) \otimes \underline{\text{id}} + ((m+1, m) \cdots (2, 1)) \cdot (\underline{f} \otimes \underline{\text{id}}^{\otimes m}) \quad \text{(by induction)} \\ &= \rho^m(f \# 1) \otimes \underline{\text{id}} + \underline{R}_1^{(2)} \otimes \cdots \otimes \underline{R}_m^{(2)} \otimes R_m^{(1)} \cdots R_1^{(1)} \cdot \underline{f} \quad \text{(by (**))} \\ &= \sum_{j=0}^{m-1} \underline{R}_1^{(2)} \otimes \cdots \otimes \underline{R}_j^{(2)} \otimes R_j^{(1)} \cdots R_1^{(1)} \cdot \underline{f} \otimes \underline{\text{id}}^{\otimes m-j} \\ & \quad + \underline{R}_1^{(2)} \otimes \cdots \otimes \underline{R}_m^{(2)} \otimes R_m^{(1)} \cdots R_1^{(1)} \cdot \underline{f} = \rho^{m+1}(f \# 1) \quad \text{(by (1))}. \end{aligned}$$

We are ready to prove the R -centralizer theorem; let us recall some facts. If A is a kS_m module algebra and $t \in \sum_{\sigma \in S_m} \sigma$, then $t \cdot A \subset A^{kS_m}$. We say A has an element of trace 1 if there exists $a \in A$ with $t \cdot a = 1$, and then $A^{kS_m} = t \cdot A$. In particular, if $\text{char } k = 0$ and $A = \text{End}_k V$, then $t(1/m!) = 1$.

Theorem 3.8. *Let (H_1, R) be a triangular Hopf algebra, $V \in H_1 \text{ Mod}$ a finite-dimensional vector space over a field k of characteristic 0, and $m \in \mathbb{N}$. Let $\rho^m: H \rightarrow \text{End}_k V^{\otimes m}$ and $\phi: kS_m \rightarrow \text{End}_k V^{\otimes m}$ be the representations described in (3.4) and (3.5). Then*

- (1) $\text{End}_{\phi(kS_m)} V^{\otimes m} = \rho^m(H)$,
- (2) $\text{End}_{\rho^m(H)} V^{\otimes m} = \phi(kS_m)$.

Proof. The proof closely follows the proof in [Fi].

(0) Let kS_m act on $\text{End}_k V^{\otimes m}$ as in (3.6).

(1) Since there exists a trace 1 element in $\text{End}_k V^{\otimes m}$, we have $(\text{End}_k V^{\otimes m})^{kS_m} = t \cdot (\text{End}_k V^{\otimes m})$. However, $(\text{End}_k V^{\otimes m})^{kS_m} = \text{End}_{\phi(kS_m)} V^{\otimes m}$. Thus to show (1) we need to show that $t \cdot (\text{End}_k V^{\otimes m}) = \rho^m(H)$.

(3.9) Now, $\rho^m(H) \subset \text{End}_{\phi(kS_m)} V^{\otimes m}$ by [Fi, 2.1].

So we show the reverse inclusion by induction on m . For $m = 1$, $\text{End}_{\phi(kS_1)} V = \text{End}_k V = \rho(H)$. Assume the inclusion for $m - 1$. Let $a = 1 + (2, 1) + \cdots +$

$((m, m-1) \cdots (2, 1)) \in kS_m$ and $t' = \sum_{\sigma \in S_{\{2, \dots, m\}}} \sigma$. Then $t = at'$, and thus, for all $\underline{f_1} \otimes \cdots \otimes \underline{f_m} \in \text{End}_k V^{\otimes m}$, $t \cdot (\underline{f_1} \otimes \cdots \otimes \underline{f_m}) = at' \cdot (\underline{f_1} \otimes \cdots \otimes \underline{f_m}) = a \cdot (\underline{f_1} \otimes t' \cdot (\underline{f_2} \otimes \cdots \otimes \underline{f_m}))$. But $t' \cdot (\underline{f_2} \otimes \cdots \otimes \underline{f_m}) \in \rho^{m-1}(H)$ by the induction hypothesis. Hence there exists a $g = \sum b_i \# h_i \in H$ with $t' \cdot (\underline{f_2} \otimes \cdots \otimes \underline{f_m}) = \rho^{m-1}(g)$. If $\Delta^{m-2}(g) = \sum g_{(1)} \otimes \cdots \otimes g_{(m-1)}$, then the above equals

$$\begin{aligned} & a \cdot \sum (\underline{f_1} \otimes \underline{g_{(1)}} \otimes \cdots \otimes \underline{g_{(m-1)}}) \\ &= a \cdot \sum (\underline{f_1} S \underline{g_{(1)}} \underline{g_{(2)}} \otimes \underline{g_{(3)}} \otimes \cdots \otimes \underline{g_{(m+1)}}) \\ &= a \cdot \left(\sum (\underline{f_1} S \underline{g_{(1)}} \otimes \text{id}^{\otimes m-1}) (\underline{g_{(2)}} \otimes \cdots \otimes \underline{g_{(m+1)}}) \right) \\ &= a \cdot \left(\sum ((\underline{f_1} S \underline{g_{(1)}}) \otimes \text{id}^{m-1}) \rho^m(g_{(2)}) \right). \end{aligned}$$

But by Theorem 3.8 $\rho^m(g_{(2)})$ is fixed by kS_m ; hence the above equals

$$a \cdot \left(\sum \underline{f_1} S \underline{g_{(1)}} \otimes \text{id}^{m-1} \right) \rho^m(g_{(2)}) = \rho^m \sum (\underline{f_1} S \underline{g_{(1)}} \# 1) \rho^m(g_{(2)})$$

(by Lemma 3.7) and this is in $\rho(H)$, as claimed.

(2) follows as in [Fi].

Remark 3.10. It is now obvious that both the classical and the “super” double centralizer theorems are special cases of Theorem 3.8 with $H_1 = k$, $R = 1 \otimes 1$, and $(H_1, R) = k\mathbb{Z}_2$, $\frac{1}{2}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g)$, respectively.

REFERENCES

- [BR] A. Berele and A. Regev, *Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras*, Adv. Math. **64** (1987), 118–175.
- [CW] M. Cohen and S. Westreich, *From supersymmetry to quantum commutativity*, J. Algebra (to appear).
- [D] V. Drinfeld, *Quantum groups*, Proc. Internat. Congr. Math. (Berkeley), Amer. Math. Soc., Providence, RI, 1986, pp. 798–819.
- [Fi] D. Fischman, *Schur’s double centralizer theorem: a Hopf algebra approach*, J. Algebra **157** (1993), 331–340.
- [Maj1] S. Majid, *Quasitriangular Hopf algebras and Yang-Baxter equations*, Internat. J. Modern Phys. A **5** (1990), 1–91.
- [Maj2] —, *Cross products by Braided groups and Bosonization*, preprint.
- [Man] Y. Manin, *Quantum groups and non-commutative geometry*, Univ. of Montreal lectures, 1988.
- [S] M.E. Sweedler, *Hopf algebras*, Benjamin, New York, 1969.

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