GEOMETRIC PROPERTIES OF SOME SUBSPACES OF $VN(G)$

ALAIN BÉLANGER AND BRIAN E. FORREST

(Communicated by J. Marshall Ash)

ABSTRACT. Let G be a locally compact group. We show that any one of the spaces $UCB(\hat{G})$, $WAP(\hat{G})$, $AP(\hat{G})$, and $C_\delta^*(G)$ is Asplund if and only if the group G is finite. We also show that any one of the spaces $VN(G)$, $UCB(\hat{G})$, and $C_\delta^*(G)$ has the DPP if and only if the group G has an abelian subgroup of finite index.

Let π be a representation of a locally compact group G . We will denote by π_d the representation of the discrete group G_d determined by π . It is clear that the spaces $A_\pi \subset B(G)$ and $A_{\pi_d} \subset B(G_d)$ are the same (see [1] for a definition of A_π and B_π). However, B_{π_d} will be, in general, a larger space than B_π . In the case where $\pi = \lambda$, $A_\lambda = A_{\lambda_d} = A(G)$ is the Fourier algebra of G . As usual, let $C_{\lambda_d}^*(G)$ be the C^* -algebra generated by $\{\lambda_d(x) | x \in G\}$ acting on $L^2(G)$. This algebra has been studied by Lau [9] and Chou, Lau, and Rosenblatt [3], where it was denoted $C_\delta^*(G)$. We shall adopt this notation.

Let $VN(G)$ denote the dual of $A(G)$. For u in $A(G)$ and T in $VN(G)$, let $u \cdot T$ be the usual modular action: $\langle u \cdot T, v \rangle = \langle T, uv \rangle$ for all v in $A(G)$. The space of all uniformly continuous functionals on $A(G)$, denoted $UCB(\hat{G})$, is the closed linear span of $\{u \cdot T | u \in A(G), T \in VN(G)\}$. Then $UCB(\hat{G})$ is in fact a C^* -subalgebra of $VN(G)$. An element T in $VN(G)$ is said to be (weakly) almost periodic if the set $\{u \cdot T | \|u\|_{A(G)} \leq 1\}$ is (weakly) relatively compact in $VN(G)$. We denote the space of all (weakly) almost periodic functionals on $A(G)$ by $(WAP(\hat{G}))$ $AP(\hat{G})$. It is immediate to see that $C_\delta^*(G)$ is a subspace of each one of these spaces.

A C^* -algebra A is scattered if every positive functional on A is the sum of a sequence of pure functionals. Chu [4] showed that A is scattered if and only if the dual of A has the RNP (see [6] for a definition of RNP). A Banach space X such that its dual has the RNP is called an Asplund space.

If G is locally compact and abelian, then $AP(\hat{G})$, $WAP(\hat{G})$, and $UCB(\hat{G})$ can be identified via the Fourier transform with the corresponding spaces of continuous functions on $\hat{G}$. We get, moreover, that each one of these spaces

Received by the editors December 1, 1992.

1991 *Mathematics Subject Classification.* Primary 43A60, 43A65, 46B20.

Key words and phrases. Locally compact group, unitary representation, group von Neumann algebra, Radon-Nikodym property, Dunford-Pettis property.

The authors were supported in part by NSERC Canada.

is a C^* -algebra and $AP(\hat{G}) = C_\delta^*(G)$. Since $AP(\hat{G})$ can also be identified with $C(\hat{G}^{\text{ap}})$, the space of continuous functions on the almost periodic compactification of G , $AP(\hat{G})$ is scattered if and only if the compact group $\hat{G}^{\text{ap}}$ is dispersed. This, of course, happens only when $\hat{G}^{\text{ap}}$ is finite, that is, if and only if G is finite. It can be shown as easily that $WAP(\hat{G})$ and $UCB(\hat{G})$ are scattered if and only if G is finite. We will show that it is essentially the same situation for all locally compact groups. Since it is not known whether $AP(\hat{G})$ and $WAP(\hat{G})$ are always C^* -subalgebras of $VN(G)$, the question becomes: When are these spaces Asplund?

Lemma 1. *Let G be a locally compact group. Then $A(G_d) \subset B_{\lambda_d}(G) = (C_\delta^*(G))^*$.*

Proof. If $P_{\lambda_d}(G)$ denotes the set of all positive definite functions associated to λ_d , then $P_{\lambda_d}(G) = A(G) \cap P(G)$. Also, $(C_\delta^*(G))^*$ is the span of all limits of sums of elements of $P_{\lambda_d}(G)$ in the topology of pointwise convergence. Let U_α be a relatively compact neighbourhood of the identity, e , of G . Pick a symmetric neighbourhood V_α of e such that $V_\alpha^2 \subset U_\alpha$. Let χ_{V_α} be the characteristic function of V_α , and let $f_\alpha = \chi_{V_\alpha}/\mu(V_\alpha)^{1/2}$. Define $u_\alpha(x) := \langle \lambda(x)f_\alpha, f_\alpha \rangle$. Then u_α belongs to $P(G) \cap A(G)$, $u_\alpha(e) = 1$, and $\text{supp } u_\alpha \subset U_\alpha$. It follows that $\chi_{\{e\}}$ belongs to the closure of $P(G) \cap A(G)$ (in the topology of pointwise convergence). Hence $\chi_{\{e\}}$ belongs to $B_{\lambda_d}(G_d)$, but this last space is invariant by translation. Thus $\chi_{\{x\}}$ belongs to $B_{\lambda_d}(G_d)$ for every $x \in G$. But then $A(G_d) \subset B_{\lambda_d}(G_d)$, since $B_{\lambda_d}(G_d)$ is closed in $B(G_d)$ and $C_{00}(G_d) \subset B_{\lambda_d}(G_d)$.

Theorem 1. *Let G be a locally compact group. The following are equivalent:*

- (i) *The group G is finite.*
- (ii) *The C^* -algebra $UCB(\hat{G})$ is scattered.*
- (iii) *The Banach space $WAP(\hat{G})$ is Asplund.*
- (iv) *The Banach space $AP(\hat{G})$ is Asplund.*
- (v) *The C^* -algebra $C_\delta^*(G)$ is scattered.*

Proof. It is clear that (i) implies each one of the other assertions. Note that $C_\delta^*(G) \subset AP(\hat{G}) \subset WAP(\hat{G})$ [9] and $C_\delta^*(G) \subset UCB(\hat{G})$. Since closed subspaces of Asplund spaces are also Asplund, we get that each one of (i)–(iv) implies (v). There only remains to prove that (v) implies (i). Suppose that $C_\delta^*(G)$ is scattered. Then $B_{\lambda_d}(G_d)$ has the RNP. It follows from Lemma 1 that $A(G_d)$ has the RNP. But this is possible only if G is finite [12, 2].

A Banach space X with dual X^* has the DPP if and only if whenever a sequence $\{x_n\}$ is weakly null in X and a sequence $\{x_n^*\}$ is weakly null in X^* , then the sequence $\{x_n^*(x_n)\}$ is null. Lemma 1 also enables us to tie the following propositions together.

Theorem 2. *Let G be a locally compact group. The following are equivalent:*

- (i) *The von Neumann algebra $VN(G)$ has the DPP.*
- (ii) *The C^* -algebra $UCB(\hat{G})$ has the DPP.*
- (iii) *The C^* -algebra $C_\delta^*(G)$ has the DPP.*
- (iv) *The group G has an abelian subgroup of finite index.*

Proof. The equivalence of (i) and (iv) is due to Lau and Ülger [10, Theorem 4.2]. The DPP is hereditary for C^* -subalgebras by [8] and [5]; hence (i) implies (ii), which in turn implies (iii). If $C_\delta^*(G)$ has the DPP then so does $B_{\lambda_d}(G_d)$

by [8] and [5]. By Lemma 1, $A(G_d) \subset B_{\lambda_d}(G)$, which means by a standard result of Arzac [1] that $A(G_d)$ is complemented in $B_{\lambda_d}(G)$. Because DPP is hereditary for complemented subspaces, $A(G_d)$ also has the DPP. By [8] G_d is type I, finite. Thus G has an abelian subgroup of finite index [11].

REFERENCES

1. G. Arzac, *Sur l'espace de Banach engendré par les coefficients d'une représentation unitaire*, Publ. Dép. Math. (Lyon) **13** (1976), 1–101.
2. A. Bélanger and B. E. Forrest, *Geometric properties of coefficient function spaces determined by unitary representations of a locally compact group*, preprint.
3. C. Chou, A. T. M. Lau, and J. M. Rosenblatt, *Approximation of compact operators by sums of translations*, Illinois J. Math. **29** (1985), 340–350.
4. C. H. Chu, *A note on scattered C^* -algebras*, J. London Math. Soc. (2) **24** (1981), 533–536.
5. C. H. Chu and B. Iochum, *The Dunford-Pettis property in C^* -algebras*, Studia Math. **97** (1990), 59–64.
6. J. Diestel and J. J. Uhl, *Vector measures*, Math. Surveys Monographs, vol. 15, Amer. Math. Soc., Providence, RI, 1977.
7. P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France **92** (1964), 181–236.
8. M. Hamana, *On linear topological properties of some C^* -algebras*, Tôhoku Math. J. **29** (1977), 157–163.
9. A. T. M. Lau, *The second conjugate algebra of the Fourier algebra of a locally compact group*, Trans. Amer. Math. Soc. **267** (1981), 53–63.
10. A. T. M. Lau and A. Ülger, *Some geometric properties on the Fourier and Fourier-Stieltjes algebras of locally compact groups, Arens regularity and related problems*, preprint.
11. T. W. Palmer, *Classes of nonabelian, noncompact, locally compact groups*, Rocky Mountain J. Math. **8** (1978), 683–741.
12. K. Taylor, *Geometry of the Fourier algebras and locally compact groups with atomic representations*, Math. Ann. **262** (1983), 183–190.

DEPARTMENT OF MATHEMATICS AND STATISTICS, YORK UNIVERSITY, NORTH YORK, ONTARIO, CANADA M3J 1P3

E-mail address: belanger@nexus.yorku.ca

DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF WATERLOO, WATERLOO, ONTARIO, CANADA N2L 3G1

E-mail address: beforrest@poppy.uwaterloo.ca