FREE NORMAL COMPLEMENTS AND THE UNIT GROUP OF INTEGRAL GROUP RINGS

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ABSTRACT. It is shown that there are only four finite groups G with the property that G has a nonabelian free normal complement in the unit group of the integral group ring ZG .

For finite commutative groups it is well known that the trivial units have a free abelian (normal) complement in the unit group of their integral group ring. Hence, it is natural to state the following problem: characterize the finite groups G which have a nonabelian free normal complement in the unit group of the integral group ring ZG . This question was posed to the author by M. M. Parmenter.

In this paper it is shown that the only groups with this property are:

1. $S_3 = \langle a, b | a^3 = 1, b^2 = 1, ba = a^{-1}b \rangle$, the symmetric group of degree 3;
2. $D_4 = \langle a, b | a^4 = 1, b^2 = 1, ba = a^{-1}b \rangle$, the dihedral group of order 8;
3. $T = \langle a, b | a^6 = 1, b^2 = a^3, ba = a^5b \rangle$, the dicyclic group of order 12; and
4. $P = \langle a, b | a^4 = 1, b^4 = 1, aba^{-1}b^{-1} = a^2 \rangle$, a group of order 16.

Note that in [4, 6, 9] it has been shown these groups indeed have the property under review.

Throughout G denotes a finite group. In the first two lemmas we will make use of the following observation. Suppose N is a nonabelian free normal complement of the trivial units in $\mathcal{U}(ZG)$. Then, since $\mathcal{U}(ZG)/N \cong \pm G$, in particular the factor group is finite, and because N is nonabelian free, it follows that $\mathcal{U}(ZG)$ does not contain a free abelian subgroup of rank two.

Lemma 1. *If $\pm G$ has a nonabelian free normal complement in $\mathcal{U}(ZG)$, then the order of an element in G is either 1, 2, 3, 4, 5, 6, 8, or 12.*

Proof. Let $g \in G$ have order n , and let $\langle g \rangle$ denote the cyclic subgroup generated by g . Since $\mathcal{U}(Z\langle g \rangle)$ does not contain a free abelian subgroup of rank 2, it follows from Higman's result (cf. [15, Theorem II.3.1]) that the rank r of

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the free abelian complement of the trivial units in $\mathcal{U}(\mathbb{Z}\langle g \rangle)$ is

$$(*) \quad r = \frac{1}{2}(n + 1 + n_2 - 2c) = 0 \text{ or } 1,$$

where c is the number of cyclic subgroups of $\langle g \rangle$ and n_2 is the number of elements of order 2 in $\langle g \rangle$.

The case $r = 0$ is well known and yields $n = 1, 2, 3, 4$, or 6 . Assume now $r = 1$, i.e., $n + n_2 - 2c - 1 = 0$. Now let p be a prime number and $\alpha > 1$ with $p^\alpha | n$. If $p \neq 2$ and $3 \neq p^\alpha$, then the above applied to the element g^{n/p^α} yields

$$p^\alpha - 2(\alpha + 1) - 1 = 0.$$

So $p^\alpha = 2\alpha + 3$, and thus $p^\alpha = 5$. It is easily verified that the identity $(*)$ fails for n divisible by 24, 16, 15, or 10, i.e., in all cases the unit group contains a free abelian subgroup of rank at least 2. Hence the only remaining possible values for n are 5, 8, and 12. The result follows. $\square$

Lemma 2. *If $\pm G$ has a nonabelian free normal complement in $\mathcal{U}(\mathbb{Z}G)$, then $\mathbb{Q}G$ has a simple Wedderburn component which is isomorphic with $M_2(\mathbb{Q})$, a two-by-two matrix ring over the rationals, and all the other simple Wedderburn components are division rings.*

Proof. Let S be a simple component of $\mathbb{Q}G$. We first show that if $S \cong M_n(D)$, D a division ring, and $n \geq 2$, then $n = 2$. Indeed suppose $n \geq 3$, and let $E_{i,j}$, $1 \leq i, j \leq n$, be a set of matrix units for S . So $E_{i,j}$ corresponds with a matrix which has 1 at the (i, j) -entry and zeros elsewhere. Since each $E_{i,j} \in \mathbb{Q}G$, there exists a positive integer m such that $mE_{i,j} \in \mathbb{Z}G$. Clearly $(mE_{1,2})^2 = (mE_{1,n})^2 = 0$. It follows easily that $1 + mE_{1,2}$ and $1 + mE_{1,n}$ generate a free abelian group of rank two, a contradiction. So $n = 2$.

Next we show that $\mathbb{Q}G$ has at most one simple Wedderburn component which is not a division ring. Suppose there are two components S_1 and S_2 which are not division rings. As above, in each $S_i \cap \mathbb{Z}G$ there exist $0 \neq \alpha_i$ with $\alpha_i^2 = 0$. It again follows that $1 + \alpha_1$ and $1 + \alpha_2$ generate a free abelian group of rank 2, a contradiction.

Third we show that if a simple component $S \cong M_2(D)$, D a division ring, then $D = \mathbb{Q}$. If not, let $d \in (D \setminus \mathbb{Q})$. Then, with the above notation, let m be a positive integer such that $mE_{1,2}, mdE_{1,2} \in \mathbb{Z}G$. As before, the units $1 + mdE_{1,2}$ and $1 + mE_{1,2}$ yield a contradiction.

So far we have shown that there is at most one component which is a matrix ring, and if such a component exists then it is isomorphic with $M_2(\mathbb{Q})$. It therefore remains to be shown that there exists a simple component which is a matrix ring. Assume the contrary, i.e., $\mathbb{Q}G$ is a direct sum of division rings. So $\mathbb{Q}G$ has no nonzero nilpotent elements. Theorem VI.1.17 in [15] therefore implies that G is either abelian or Hamiltonian of order $2^n m$, m odd such that the multiplicative order of 2 modulo m is odd. The former case is excluded as then $\mathcal{U}(\mathbb{Z}G)$ is abelian, and no nonabelian free subgroup exists. In the latter case, it follows from the results of Moser [8] and Fein, Gordon, and Smith [2] (see also Proposition VI.1.13 and Theorem VI.1.15 in [15]) that if a prime p divides m , then p is congruent to 1 or 7 modulo 8; hence, $p \geq 7$. Lemma 1 yields $m = 1$. Hence G is a Hamiltonian 2-group, and thus by Higman's theorem, the units of $\mathbb{Z}G$ are trivial. This contradicts with the existence of a noncommutative free normal complement. $\square$

The next two lemmas deal with proper subgroups of G .

Lemma 3. *Suppose $\mathbf{Q}G$ has at most one simple component which is not a division ring, and suppose that if such a component exists then it is isomorphic with $M_2(\mathbf{Q})$. If S is a proper subgroup of G , then $\mathbf{Q}S$ is a sum of division rings and S is either abelian or a Hamiltonian group. If, moreover, S is nonabelian and every element of S has order 1, 2, 3, 4, 5, 6, 8, or 12, then S is a Hamiltonian 2-group.*

Proof. In the last part of the proof of Lemma 2 it is shown that if S is non-abelian and $\mathbf{Q}S$ is a sum of division rings then S is Hamiltonian. Furthermore, S is a Hamiltonian 2-group if every element has order 1, 2, 3, 4, 5, 6, 8, or 12.

So the lemma will follow if we show that it is impossible to write $\mathbf{Q}S = C \oplus B$, where B is a sum of simple rings, $C = \mathbf{Q}Sf \cong M_n(D)$, $n > 1$, D a division ring, and f a primitive central idempotent of $\mathbf{Q}S$. Suppose by contradiction that such a decomposition exists. For any primitive central idempotent e of $\mathbf{Q}G$ either $ef = 0$ or $ef \neq 0$. Assume $ef \neq 0$; then the natural ring homomorphism

$$C \rightarrow \mathbf{Q}Ge: \alpha \mapsto \alpha e$$

is injective (as C is simple). So $\mathbf{Q}Ge$ contains nilpotent elements, and thus, by the assumption, $\mathbf{Q}Ge \cong M_2(\mathbf{Q})$. It follows that

$$\mathbf{Q}Ge = Ce = \mathbf{Q}Sfe.$$

So, fe is the right identity of $\mathbf{Q}Ge$. Hence $e = fe$. We have shown that if $ef \neq 0$, then $ef = e$ and $\mathbf{Q}Ge = Ce \cong M_2(\mathbf{Q})$. As $\mathbf{Q}Ge$ has exactly one simple component which is isomorphic with $M_2(\mathbf{Q})$ there is only one e with $ef \neq 0$. Since $f = \sum ef$, where the sum runs over all the primitive central idempotents of $\mathbf{Q}G$, we obtain $f = ef = e$. Consequently $\mathbf{Q}Sf = \mathbf{Q}Gf$, and thus $Gf \subseteq \mathbf{Q}S$. However, this is impossible as S is a proper subgroup of G . $\square$

Lemma 4. *Suppose $\mathbf{Q}G$ has at most one simple component which is not a division ring, and suppose that if such a component exists then it is isomorphic with $M_2(\mathbf{Q})$. If M is a maximal proper normal subgroup of G , then G/M is a cyclic group of prime order.*

Proof. The simple components of $\mathbf{Q}(G/M)$ are simple components of $\mathbf{Q}G$. Hence $\mathbf{Q}(G/M)$ has at most one simple component which is not a division ring, and if so this component is isomorphic with $M_2(\mathbf{Q})$.

It is sufficient to show that the simple group $S = G/M$ is commutative. Suppose not; then S contains an element of order 2, say s . Since $S' = S$, Lemma 1.2 in [1] yields

$$\mathbf{Q}S \cong \mathbf{Q}(S/S) \oplus \omega(S) \cong \mathbf{Q} \oplus \omega(S),$$

where $\omega(S)$ is the augmentation ideal and a sum of noncommutative simple rings. Since S is embedded in each simple component $\mathbf{Q}Se$ of $\omega(S)$, e a primitive central idempotent, it follows that $e(1+s) \neq 0$, $e(1-s) \neq 0$, and $e(1+s)(1-s) = 0$. So every simple component of $\omega(S)$ has nontrivial zero divisors and therefore is not a division ring. Hence there can be only one

such simple component. So $\omega(S) \cong M_2(\mathbf{Q})$. Counting dimensions we obtain $|S| = 1 + \dim(\omega(S)) = 5$. So $S \cong \mathbf{Z}_5$, a contradiction. $\square$

We now show that elements of order 5 also are excluded in case the trivial units have a nonabelian free normal complement in the unit group. Furthermore, we exclude some division rings. By ξ_8 we denote a primitive root of unity of order 8.

Lemma 5. *If $\pm G$ has a nonabelian free normal complement in $\mathcal{U}(\mathbf{Z}G)$, then G has no elements of order 5. Furthermore, if a division ring D is a simple component of $\mathbf{Q}G$, then D is neither $\mathbf{Q}(\xi_8)$ nor a quaternion algebra over $\mathbf{Q}(\sqrt{2})$.*

Proof. Because of Lemma 2, $\mathbf{Q}G$ has exactly one simple component which is not a division ring, and this simple component is isomorphic with $M_2(\mathbf{Q})$.

Let M be a maximal proper normal subgroup of G . By Lemma 4, $G = \langle M, g \rangle$, $g^p \in M$, p a prime number.

We first show that 5 does not divide the order of M . Suppose this is false, then clearly M is not a Hamiltonian 2-group. So, by Lemma 3, M is abelian. Moreover, Lemma 1, $M \cong C_5 \times \cdots \times C_5$, a direct product of copies of the cyclic group C_5 of order 5. Hence,

$$\mathbf{Q}M \cong \mathbf{Q} \oplus \omega(M),$$

and $\omega(M)$ is a direct sum of field extensions of degree 4 over $\mathbf{Q}$ (see, for example, [15, Proposition II.2.6]). The rational component corresponds with the primitive central idempotent $f = \frac{1}{|M|} \sum_{m \in M} m$. Note that f is also central in $\mathbf{Q}G$ as M is a normal subgroup. Let e be the primitive central idempotent of $\mathbf{Q}G$ such that $\mathbf{Q}Ge \cong M_2(\mathbf{Q})$. As $\mathbf{Q}Ge$ is noncommutative of dimension 4 over the rationals, it does not contain a field extension of degree 4 over $\mathbf{Q}$. Hence $\omega(M)e = 0$. So

$$\mathbf{Q}Ge = \sum_{1 \leq i \leq p} (\mathbf{Q}f g^i e).$$

It follows that $\mathbf{Q}Ge$ is commutative, a contradiction. Hence we have shown that 5 does not divide $|M|$.

Second, we show that 5 does not divide $|G/M|$. Since $G' \subseteq M$, it follows from the first part that it is sufficient to show that 5 does not divide $|G/G'|$. Suppose the contrary. Again

$$\mathbf{Q}G \cong \mathbf{Q}(G/G') \oplus \omega_G(G'),$$

where $\omega_G(G')$ is the augmentation ideal determined by the normal subgroup G' . As $\mathbf{Q}(G/G')$ is abelian, it follows that $\omega_G(G')$ has one simple component isomorphic with $M_2(\mathbf{Q})$. Hence, as before, there exists a nonperiodic unit element in $\mathbf{Z}G$ that is of the form $1 + \alpha$, where $\alpha \in \mathbf{Q}Ge$, and $\alpha^2 = 0$. Also, as $|G/G'|$ is divisible by 5, it follows from Higman's results (cf. [15, Theorem II.3.1]) that $\mathcal{U}(\mathbf{Z}(G/G'))$ contains a unit $\bar{u}$ of infinite order. Let u be a preimage of $\bar{u}$ in $\mathbf{Z}G$, and let $\widehat{G'} = \frac{1}{|G'|} \sum_{g \in G'} g$. Then $\omega_G(G') = \mathbf{Q}G(1 - \widehat{G'})$ and

$$v = u\widehat{G'} \oplus (1 - \widehat{G'})$$

is a unit in $\mathbf{Z}G\widehat{G'} \oplus \mathbf{Z}G(1 - \widehat{G'})$. Since $\mathbf{Z}G \subseteq \mathbf{Z}G\widehat{G'} \oplus \mathbf{Z}G(1 - \widehat{G'})$ are $\mathbf{Z}$ -orders, it follows from [15, (2.9), p. 49] that v^k , for some $k \geq 1$, is a nonperiodic unit

in ZG . It is easily verified that v^k and $1 + \alpha$ generate a free abelian group of rank two, a contradiction.

Finally, assume that e is a primitive central idempotent such that $\mathbb{Q}Ge = D$ is either a quaternion algebra over $\mathbb{Q}(\sqrt{2})$ or $\mathbb{Q}(\xi_8)$. Then, $Z(D)$ contains a $\mathbb{Z}$ -order (either $\mathbb{Z}[\sqrt{2}]$ or $\mathbb{Z}[\xi_8]$) which contains an element u of infinite order. As in the previous argument, for some $k > 0$, $(1 - e) + u^k$ is a unit of infinite order in ZG ; again yielding a contradiction. $\square$

Next we will show that, under the previously considered assumptions, G has a proper normal subgroup of index 2 in G . But first we need the following lemma, which is a consequence of [3, Theorem 2, p. 59]. A proof (in a more general situation) can be found in [5].

Lemma 6. *Let e be a primitive central idempotent of $\mathbb{Q}G$ such that $\mathbb{Q}Ge \cong M_2(\mathbb{Q})$. If f is a noncentral idempotent in $\mathbb{Q}Ge$, then there exist matrix units $E_{u,v}$, $1 \leq u, v \leq 2$, such that $f = E_{1,1}$, and $\mathbb{Q}Ge$ can be considered as a matrix ring over $\mathbb{Q}$ which is the centralizer of all $E_{u,v}$. $\square$*

Lemma 7. *Suppose $\mathbb{Q}G$ has exactly one simple component which is not a division ring, and suppose this simple component is isomorphic with $M_2(\mathbb{Q})$. If $x \in G$ and M is a maximal proper normal subgroup of G with $x^3 = 1$ and $G = \langle M, x \rangle$, then the cyclic group $\langle x \rangle$ is normal in G .*

Proof. It is sufficient to show that the idempotent $f = \frac{1}{3}(1 + x + x^2)$ is central. To do so we show that for every central primitive idempotent $e \in \mathbb{Q}G$, either $ef = 0$ or $ef = f$. As ef is an idempotent the latter is clear when $\mathbb{Q}Ge$ is a division ring. So we are left with the case that $\mathbb{Q}Ge \cong M_2(\mathbb{Q})$. If ef is central in the simple algebra $\mathbb{Q}Ge$ then obviously $ef = 0$ or $ef = e$. So suppose that ef is not central. By Lemma 6, $\mathbb{Q}Ge$ contains matrix units $E_{1,1}, E_{1,2}, E_{2,1}$, and $E_{2,2}$ with $E_{1,1} = ef$ and $E_{2,2} = e - ef$. As $x = fxf + (1 - f)x(1 - f)$, we obtain

$$ex = efxf + (e - ef)x(e - ef).$$

So ex corresponds with a matrix of the type $\begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix}$, $s, t \in \mathbb{Q}$. Since $x^3 = 1$, it follows that $s^3 = t^3 = 1$. Therefore, $s = t = 1$. Hence $ex = e$, and $M_2 \cong \mathbb{Q}Ge$ is a homomorphic image of $\mathbb{Q}M$. However, this is impossible because of Lemma 3; the result follows. $\square$

Lemma 8. *Suppose $\mathbb{Q}G$ has exactly one simple component which is not a division ring, and suppose this simple component is isomorphic with $M_2(\mathbb{Q})$. If every element of G has order 1, 2, 3, 4, 6, 8, or 12, then G has a maximal proper normal subgroup M such that G/M is a cyclic group of order 2.*

Proof. First of all notice that G is a group of even order, as $\mathbb{Q}G$ contains $M_2(\mathbb{Q})$ as a simple component.

Because of the assumptions and Lemma 4, G has a maximal proper normal subgroup M such that $G/M \cong C_2$ or $G/M \cong C_3$. In the former case the result follows. So assume that $G = \langle M, x \rangle$, with $x^3 = 1$, $x^6 = 1$, or $x^{12} = 1$. Replacing, if necessary, x by x^2 or x^4 we may assume that $x^3 = 1$. By Lemma 7, $\langle x \rangle$ is a normal subgroup of G . As $\mathbb{Q}(G/\langle x \rangle)$ is a homomorphic image of $\mathbb{Q}G$, one obtains that $\mathbb{Q}(G/\langle x \rangle)$ has at most one simple component which is not a division ring, and if such a component exists then it is isomorphic with

$M_2(\mathbb{Q})$. If $\mathbb{Q}(G/\langle x \rangle)$ has no matrix-component, then, as in the first part of the proof of Lemma 3, $G/\langle x \rangle$ is Abelian or a Hamiltonian 2-group. Consequently there exist a maximal proper normal subgroup N of G containing $\langle x \rangle$ such that $G/N \cong C_2$. On the other hand, if $\mathbb{Q}(G/\langle x \rangle)$ contains $M_2(\mathbb{Q})$ as a simple component, then $|G/\langle x \rangle| < |G|$ and the result follows by induction on the order of the group. $\square$

Now we show that G is generated by two elements in case $\mathbb{Q}G$ has the appropriate Wedderburn decomposition.

Lemma 9. *Suppose $\mathbb{Q}G$ contains a primitive central idempotent e such that $\mathbb{Q}Ge \cong M_2(\mathbb{Q})$. Further assume that every other simple component of $\mathbb{Q}G$ is a division ring. If $a, b \in G$ are such that ae and be do not commute, then $G = \langle a, b \rangle$.*

Proof. Let F be the $\mathbb{Q}$ -subalgebra of $\mathbb{Q}Ge$ generated by be , and let H be the cyclic subgroup of Ge generated by be . Then F is a homomorphic image of the commutative semisimple Artinian group algebra $\mathbb{Q}H$. Hence $F = F_1 \oplus \cdots \oplus F_k$, a direct sum of fields. Let f_i denote the identity of F_i . Note that $e = f_1 + \cdots + f_k$. We consider two cases.

Case 1: $k \geq 2$. Then $f_1 \neq 0$, $f_1 \neq e$, and thus f_1 is not central in $\mathbb{Q}Ge$. Hence by Lemma 6, $\mathbb{Q}Ge$ has a set of matrix units $E_{1,1}, E_{1,2}, E_{2,1}, E_{2,2}$ with $f_1 = E_{1,1}$ and $f_2 = E_{2,2}$ (note that $k = 2$ follows and also $F_1 \cong \mathbb{Q} \cong F_2$). With respect to these matrix units the elements of F are represented by matrices of the form $\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$, $x, y \in \mathbb{Q}$. Further let ae be represented by the matrix $\begin{bmatrix} u & v \\ s & t \end{bmatrix}$, where $u = f_1 a f_1 e$, $v = f_1 a f_2 e$, $s = f_2 a f_1 e$, and $t = f_2 a f_2 e$. Now, $v \neq 0$ or $s \neq 0$ because otherwise ae and be commute. Hence the $\mathbb{Q}$ -subalgebra of $\mathbb{Q}Ge$ generated by ae and be contains nilpotent elements. As $\mathbb{Q}Ge$ is 4-dimensional, it follows that this subalgebra equals $\mathbb{Q}Ge$. Consequently, $\mathbb{Q}Ge$ is an epimorphic image of the group algebra $\mathbb{Q}\langle a, b \rangle$. Lemma 3 yields that $G = \langle a, b \rangle$.

Case 2: $k = 1$. So F is a field. Note that F is at least 2-dimensional as a $\mathbb{Q}$ -vector space, because be is not central and therefore does not belong to $\mathbb{Q}e$. Also, as ae and be do not commute, $ae \notin F$. So $\dim_{\mathbb{Q}}(\mathbb{Q}\langle a, b \rangle e) \geq 4$. Hence $\mathbb{Q}\langle a, b \rangle e = \mathbb{Q}Ge \cong M_2(\mathbb{Q})$. Again, by Lemma 3, it follows that $\langle a, b \rangle = G$. $\square$

Theorem 10. *Let G be a finite group. The following conditions are equivalent:*

- (1) $\pm G$ has a nonabelian free normal complement in $\mathcal{U}(\mathbb{Z}G)$.
- (2) Every element of G has order 1, 2, 3, 4, 6, 8, or 12 and $\mathbb{Q}G \cong M_2(\mathbb{Q}) \oplus (\bigoplus_i D_i)$, where each D_i is a division ring and D_i is neither $\mathbb{Q}(\xi_8)$ nor a quaternion algebra over $\mathbb{Q}(\sqrt{2})$.
- (3) G is isomorphic with either S_3 , D_4 , T , or P .

Proof. That (1) implies (2) follows from Lemmas 1, 2, and 5. In the introduction it is mentioned that (3) implies (1) is known. So we now prove (2) implies (3). By Lemma 8, let M be a maximal proper normal subgroup of G with $G/M \cong C_2$. Write $G = \langle M, x \rangle$, $x^2 \in M$. By Lemma 3, M is either abelian or a Hamiltonian 2-group.

We first deal with the case that M is abelian. Suppose $M/(Z(G) \cap M)$ is generated by the cosets $m_i(Z(G) \cap M)$, $1 \leq i \leq k$. Let e be the primitive central idempotent of $\mathbb{Q}G$ such that $\mathbb{Q}Ge \cong M_2(\mathbb{Q})$. Then, for some i , say

$i = 1$, m_1e and xe do not commute. Lemma 9 yields that $G = \langle m_1, x \rangle$. We simply denote $m = m_1$. Since M is abelian and $x^2 \in M$, $m(xmx^{-1}) = z \in (Z(G) \cap M)$ or, equivalent, $m^{-1}xm = zm^{-2}x$. If $m^2 \in Z(G)$, then $G/Z(G) \cong C_2 \times C_2$. It follows easily from the assumptions on the order of the elements and the results in [4] that $G \cong D_4$ or $G \cong P$. So for the remainder of this case we assume that $m^2 \notin Z(G)$. Let $S = \langle m^2, x \rangle$. Note that S is a nonabelian subgroup of G . Again we deal with two cases: (i) $S \neq G$ and (ii) $S = G$.

(i) $S \neq G$. Hence by Lemma 3, $S = Q_8 \times C_2 \times \cdots \times C_2$, where Q_8 is the quaternion group of order 8. Clearly $|S'| = 2$ and $S/S' \cong C_2 \times \cdots \times C_2$. Since S is generated by two elements only, $|S/S'| = 2$ or 4, and thus, because S is not abelian, $|S| = 8$. If S is normal in G (i.e., $z \in S$), then $G/S \cong C_2$ and thus $|G| = 16$. From the assumptions and the study of the groups of order 16 in [7] we obtain that $G \cong P$. On the other hand, if S is not normal in G , then $S \subset T = \langle m^2, x, z \rangle \subset G$. Hence, as in the previous argument, S and T are Hamiltonian 2-groups, $|S| = 8$, $|T| = 16$, and thus $|G| = 32$. So $S' = T'$ is normal in G . It is then shown, as in the last part of the proof for M a Hamiltonian 2-group, that this case is impossible.

(ii) $S = G$. So $m \in \langle m^2, x \rangle$. Consequently $m(Z(G) \cap M)$ has odd order in $M/(Z(G) \cap M)$. The assumptions therefore yield that m^3 is central and m has order 3, 6, or 12. It follows that $z^6 = 1$. Hence

$$x(mz)^4 = (mz)^{-4}x.$$

Replacing m^2 by $(mz)^4$ (this is allowed by Lemma 9) we may assume $G = \langle m^2, x \rangle$ with $xm^2 = m^{-2}x$, m^2 of order 3. So $\langle m^2 \rangle$ is a normal subgroup of G . The order of x is either 2, 3, 4, 6, 8, or 12. It cannot be 3, because otherwise G is a group of order 9, and thus commutative. Also the order is not 6, for else $S_3 \cong \langle x^3, m^2 \rangle$ is a proper subgroup of G , in contradiction with Lemma 3. Also order 8 is impossible. For if not, then $Q(G/\langle m^2 \rangle) \cong Q(\langle x|x^8 = 1 \rangle)$ (and thus also QG) has a simple component of the type $Q(\xi_8)$, a contradiction. Further, order 12 is also impossible, for otherwise $\langle x^3, m^2 \rangle$ is a noncommutative proper subgroup of order 12; this is impossible because of Lemma 3. On the other hand, if the order is 2, then $|G| = 6$, and $G \cong S_3$. If the order is 4, then $|G| = 12$. The only noncommutative groups of order 12 are $S_3 \times C_2$, A_4 the alternating group of degree 4, and T . It is well known that the two former cases do not have the required Wedderburn decomposition. So $G \cong T$.

This finishes the proof for M abelian.

Suppose now that m is a Hamiltonian 2-group. Then M' is a normal subgroup of G and $|M'| = 2$. Further

$$QG \cong Q(G/M') \oplus \omega_{M'}(M)QG.$$

As $\omega_M(M')$ is a direct sum of quaternion algebras over the rationals, it follows that every simple component of $\omega_{M'}(M)QG$ is not isomorphic with $M_2(\mathbb{Q})$. Hence G/M' satisfies the same assumptions as G . Since M/M' is abelian and G is a 2-group, it follows from the previous case that $|G/M'| = 8$ or 16. So $|G| = 16$ or $|G| = 32$. The groups of order 16 have been studied in [7], and it follows that $G \cong P$ if $|G| = 16$. We now finish the proof by showing that $|G| = 32$ is impossible. Suppose not; then $G/M' \cong P = \langle a, b | a^4 = 1, b^4 = 1,$

$aba^{-1}b^{-1} = a^2\}$. The elements of order 2 in P are a^2, b^2, a^2b^2 . However, $M/M' \cong C_2 \times C_2 \times C_2$ has 7 elements of order 2, a contradiction. $\square$

During the past few years there has been a lot of progress on determining finitely many generators of a subgroup of finite index of the unit group (see, for example, [5, 10, 12, 13, 14]). In particular, subgroups of finite index are a bit better understood. Hence, one might ask for a characterization of those finite groups G which are such that $\mathcal{U}(ZG)$ has a nonabelian free subgroup of finite index. Obviously, if G is such a group then the unit group still does not contain a free abelian subgroup of rank 2. Therefore, our proofs remain valid in this situation. So S_3, D_4, T , and P are the only groups having this property.

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