A CONNECTION BETWEEN WEAK REGULARITY AND THE SIMPLICITY OF PRIME FACTOR RINGS

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Dedicated to the memory of Professor Pere Menal

ABSTRACT. In this paper, we show that a reduced ring R is weakly regular (i.e., $I^2 = I$ for each one-sided ideal I of R) if and only if every prime ideal is maximal. This result extends several well-known results. Moreover, we provide examples which indicate that further generalization of this result is limited.

Throughout this paper R denotes an associative ring with identity. All prime ideals are assumed to be proper. The prime radical of R and the set of nilpotent elements of R are denoted by $P(R)$ and $N(R)$, respectively. The connection between various generalizations of von Neumann regularity and the condition that every prime ideal is maximal will be investigated. This connection has been investigated by many authors [2, 3, 5, 7, 12, 14]. The earliest result of this type seems to be by Cohen [3, Theorem 1]. Storrer [12] was able to provide the following result: If R is a commutative ring then the following are equivalent: (1) R is π -regular; (2) $R/P(R)$ is regular; and (3) all prime ideals of R are maximal ideals. Fisher and Snider extended this result to P.I. rings [5, Theorem 2.3]. On the other hand, Chandran generalized Storrer's result to duo rings [2, Theorem 3]. Next Hirano generalized Chandran's result to right duo rings [7, Corollary 1]. More recently the result was generalized to bounded weakly right duo rings by Yao [14, Theorem 3].

As a corollary of our main result, we show that if $R/P(R)$ is reduced (i.e., $N(R) = P(R)$) then the following are equivalent: (1) $R/P(R)$ is weakly regular; (2) $R/P(R)$ is right weakly π -regular; and (3) every prime ideal of R is maximal. This result generalizes Hirano's result for right duo rings. A further consequence of our main result is that if R is reduced then R is weakly regular if and only if every prime factor ring of R is a simple domain. This result can be compared to the well-known fact that when R is reduced, then R is von Neumann regular if and only if every prime factor ring of R is a division ring. We conclude our paper with some examples which illustrate and delimit our results.

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Definition 1. (1) A ring R is *right (left) weakly regular* if $I^2 = I$ for each right (left) ideal I of R , equivalently $x \in xRxR$ ($x \in RxRx$) for every $x \in R$. R is *weakly regular* if it is both left and right weakly regular [11]. Note right (left) weakly regular rings are also called right (left) fully idempotent rings.

(2) A ring R is called *π -regular* if for every $x \in R$ there exists a natural number $n = n(x)$, depending on x , such that $x^n \in x^n Rx^n$.

(3) A ring R is *right (left) weakly π -regular* if for every $x \in R$ there exists a natural number $n = n(x)$, depending on x , such that $x^n \in x^n Rx^n R$ ($x^n \in Rx^n Rx^n$). R is *weakly π -regular* if it is both right and left weakly π -regular [6]. Every π -regular ring, biregular ring (including simple rings), and right duo ring satisfying d.c.c. on principal ideals is right weakly π -regular.

Definition 2. A ring R is *2-primal* if and only if $P(R) = N(R)$ [1].

Definition 3. An ideal I of R is *completely prime* if $xy \in I$ implies either $x \in I$ or $y \in I$ where $x, y \in R$. Also I is called *completely semiprime* if $x^2 \in I$ implies $x \in I$.

Lemma 4. A ring R is 2-primal if and only if every minimal prime ideal is completely prime.

Proof. See [13, Proposition 1.11].

Lemma 5. If R is a 2-primal ring and $R/P(R)$ is right weakly π -regular, then every prime ideal of R is maximal.

Proof. Let P be a prime ideal of R . Then there exists a minimal prime ideal $X \subseteq P$ which is completely prime by Lemma 4. Let $\bar{R} = R/X$. Then $\bar{R}$ is a right weakly π -regular domain. Let a be a nonzero element in $\bar{R}$. There exists a positive integer k such that $a^k(y - 1) = 0$, where $y \in \bar{R}a^k\bar{R}$. Hence R/X is a simple ring. Thus X is a maximal ideal and so is P .

A ring R is called *strongly π -regular* if for every x in R there exists a natural number $n = n(x)$, depending on x , such that $x^n R = x^{n+1} R$. By Dischinger [4], this condition is left-right symmetric.

From Lemma 5 and [5], we obtain the following result which appeared in [7].

Corollary 6. Let R be a 2-primal ring. Then the following conditions are equivalent:

- (1) R is strongly π -regular.
- (2) R is π -regular.
- (3) Every prime factor ring of R is a division ring.

Proof. Obviously (1) implies (2). Also by Lemma 5, (2) implies (3). Now by [5], since every prime factor ring is strongly π -regular by condition (3), R is strongly π -regular. Thus (3) implies (1).

Lemma 7. If S is a completely semiprime ideal of R and $x_1 x_2 \cdots x_n \in S$, then $x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)} \in S$, where σ is any permutation of $\{1, \dots, n\}$.

Proof. See [9] and [10].

Recall that a ring is *reduced* if there is no nonzero nilpotent element.

Theorem 8. *Let R be a reduced ring. Then the following conditions are equivalent:*

- (1) R is weakly regular.
- (2) R is right weakly π -regular.
- (3) Every prime ideal of R is maximal.

Proof. Clearly (1) implies (2). Lemma 5 shows that (2) implies (3). So we will assume that every prime ideal of R is maximal and show that R is weakly regular. Observe that in a reduced ring every minimal prime ideal is completely prime by Lemma 4 or [8]. Now since every prime ideal is maximal, then every prime ideal is completely prime. Let a be any nonzero element in R . If $RaR = R$, then $a = a1 \in aRaR$. Thus we may assume $RaR \neq R$. Then RaR is contained in a maximal ideal which is also a prime ideal. Let T be the union of all prime ideals which contain a . Let $S = R \setminus T$. Since every prime ideal is completely prime, S is a multiplicatively closed set. Let F be the multiplicatively closed system generated by the set $\{a\} \cup S$.

Now we assert that $0 \in F$. Suppose this were not true, then partial order the collection of ideals disjoint with F by set inclusion. By Zorn's lemma, we get an ideal M which is maximal disjoint with F . Then M is a prime ideal and so a maximal ideal by hypothesis. Since $a \notin M$, there exist $p \in M$ and $c \in RaR$ such that $p + c = 1$. It follows that $p \notin T$. Thus $p \in S \subseteq F$, which implies $p \in F \cap M = \emptyset$, a contradiction. Thus $0 \in F$, so

$$0 = a^{n_1}s_1a^{n_2}s_2 \cdots a^{n_t}s_t,$$

where $s_i \in S$, and we may assume the integers $n_1, n_2, \dots, n_t$ are positive. Then, using Lemma 7 and the fact that R is reduced, there exists $s \in S$ such that $as = 0$. Observe a proper ideal cannot contain both a and s (otherwise a prime ideal would contain both of them which would contradict the definition of S and T). Hence $RaR + RsR = R$. Let $a_0 \in RaR$ and $s_0 \in RsR$ such that $a_0 + s_0 = 1$. Therefore $aa_0 + as_0 = a$. Now using Lemma 7 and the fact that $\{0\}$ is a completely semiprime ideal, $aRsR = 0$. Thus $a = aa_0 \in aRaR$. Similarly, $a = a_0a \in RaRa$. Consequently R is weakly regular.

Corollary 9. *Let R be a 2-primal ring. The following conditions are equivalent:*

- (1) $R/\mathbf{P}(R)$ is weakly regular.
- (2) $R/\mathbf{P}(R)$ is right weakly π -regular.
- (3) Every prime ideal of R is maximal.

Proof. Clearly (1) implies (2). Lemma 5 shows that (2) implies (3). So we will assume that every prime ideal of R is maximal. Then every prime ideal of $R/\mathbf{P}(R)$ is maximal. Since R is 2-primal, $R/\mathbf{P}(R)$ is reduced. Thus by Theorem 8, $R/\mathbf{P}(R)$ is weakly regular.

Corollary 10 [7, Corollary 1]. *Let R be a right (or left) duo ring. Then R is π -regular if and only if every prime ideal is maximal.*

Proof. Since any prime ideal is completely prime in a right (or left) duo ring, R is 2-primal. Assume R is π -regular. Then R is weakly π -regular, and so $R/\mathbf{P}(R)$ is weakly π -regular. Thus by Corollary 9, every prime ideal is maximal.

Conversely, by Corollary 9, $R/\mathbf{P}(R)$ is weakly regular. Since R is right duo, $R/\mathbf{P}(R)$ is strongly regular. Then each prime factor ring of R is also strongly

regular. Hence by Theorem 2.1 of [5], R is both left and right π -regular. Consequently R is a π -regular ring.

Hirano [7] has shown that for a P.I. ring the concepts of right weak π -regularity, π -regularity, and strong π -regularity are equivalent. With this in mind, one can see that Corollary 9 is analogous to Theorem 2.3 of [5]. However these results are distinct in that Corollary 9 can be applied to a simple domain which is not a division ring and hence is not a P.I. ring. When R is reduced, it is well known that R is von Neumann regular if and only if every prime factor ring of R is a division ring. It is interesting to compare this fact with the following corollary.

Corollary 11. *Let R be a reduced ring. Then R is right (and so left) weakly regular if and only if every prime factor ring of R is a simple domain.*

Finally we provide two examples.

Example 12. By Fisher and Snider [5], it was shown that a ring R is strongly π -regular if and only if $R/\mathbf{P}(R)$ is strongly π -regular. But this fact does not hold for the case when R is right weakly π -regular. Indeed there is a 2-primal ring R such that $R/\mathbf{P}(R)$ is weakly regular but R is neither left nor right weakly π -regular. Assume that $W_1[F]$ is the first Weyl algebra over a field F of characteristic zero. Recall $W_1[F] = F[\mu, \lambda]$, the polynomial ring with indeterminates μ and λ with $\lambda\mu = \mu\lambda + 1$. Then $W_1[F]$ is a simple Noetherian domain. Now let R be the ring

$$\begin{pmatrix} W_1[F] & W_1[F] \\ 0 & W_1[F] \end{pmatrix}.$$

Consider the following element in R :

$$a = \begin{pmatrix} \mu & \lambda \\ 0 & 0 \end{pmatrix}.$$

Then it can be easily checked that a^k is not in $a^k R a^k R$ for any positive integer k . So R is not right weakly π -regular. Also it can be checked that R is not left weakly π -regular.

Now the prime radical $\mathbf{P}(R)$ of R is

$$\begin{pmatrix} 0 & W_1[F] \\ 0 & 0 \end{pmatrix}.$$

So $R/\mathbf{P}(R) \cong W_1[F] \oplus W_1[F]$, which is weakly regular, and hence it is weakly π -regular. Furthermore we may check that $\mathbf{P}(R) = \mathbf{N}(R)$ and hence R is a 2-primal ring.

Example 13. In Corollary 9, the condition “ R is 2-primal” is not superfluous. As in Example 12, let $W = W_1[F]$ be the first Weyl algebra over a field F of characteristic zero. Now let

$$R = \{(s_k)_{k=1}^{\infty} \mid s_k \in \text{Mat}_2(W) \text{ is eventually a constant upper triangular matrix}\},$$

which is a subring of $\Pi \text{Mat}_2(W)$, where $\text{Mat}_2(W)$ denotes the full ring of

2-by-2 matrices over W . Then it can be easily checked that the ring R is a semiprime ring. Now our claim is that every prime ideal is maximal. Let P be a prime ideal of R .

Case 1. Assume that the n th component of all elements of P is zero for some n . Let $e_n = (0, 0, \dots, 0, 1, 0, \dots)$, where 1 is in the n th component. Let $x \in R$ such that x has zero in its n th component. Then $e_n R x = 0$ and so x is in P . Therefore,

$$P = \{ (s_k)_{k=1}^{\infty} \in R \mid s_n = 0 \}$$

and thus P is maximal.

Case 2. Assume that for any n , there exists an element of P with a nonzero entry in its n th component. In this case, $\bigoplus_{k=1}^{\infty} \text{Mat}_2(W) \subseteq P$. Now let

$$I_0 = \left\{ (s_k)_{k=1}^{\infty} \in R \mid s_k = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \text{ eventually for some } b \text{ in } W \right\}.$$

Then $I_0^2 = \bigoplus_{k=1}^{\infty} \text{Mat}_2(W) \subseteq P$ and so $I_0 \subseteq P$. Let

$$e = \left\{ (s_k)_{k=1}^{\infty} \mid s_k = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ for all } k \right\}$$

and

$$f = \left\{ (s_k)_{k=1}^{\infty} \mid s_k = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ for all } k \right\}.$$

Then $e R f \subseteq I_0 \subseteq P$ and so either e is in P or f is in P . Now without loss of generality, assume that e is in P . Then

$$I_1 = \left\{ (s_k)_{k=1}^{\infty} \mid s_k = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \text{ eventually for some } a \text{ and } b \text{ in } W \right\}$$

is contained in P . So if $y \in R \setminus P$, then we may assume that $y = (y_k)_{k=1}^{\infty}$ with

$$y_k = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

eventually for some a, b , and nonzero c in W . Then in this situation, $P + RyR = R$ and so P is maximal. Furthermore, consider the element $s = (s_k)_{k=1}^{\infty}$ in R such that

$$s_k = \begin{pmatrix} \mu & \lambda \\ 0 & 0 \end{pmatrix}$$

for all k . Then for any m , $s^m \notin s^m R s^m R$ and hence R is not right weakly π -regular. Therefore, in our Corollary 9, the condition “ R is 2-primal” is not superfluous. Also note that there is a prime von Neumann regular ring which is not simple. So, in our Corollary 9, without the hypothesis “ R is 2-primal”, there is no relation between $R/\mathbf{P}(R)$ is right weakly π -regular and the fact that every prime ideal is maximal.

Recall that a ring is of *bounded index k of nilpotency* if $a^k = 0$ for every nilpotent element a . Therefore, a reduced ring is a ring of bounded index 1 of nilpotency. So one might suspect the possibility of a generalization of Theorem 8 to the case of semiprime rings of bounded index of nilpotency. But Example 13 nullifies this possibility because R is a semiprime ring of bounded index 2 of nilpotency.

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ADDED IN PROOF

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