COUNTABLE SIZE NONMETRIZABLE SPACES WHICH ARE STRATIFIABLE AND κ -METRIZABLE

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Dedicated to Professor Akihiro Okuyama on his 60th birthday

ABSTRACT. K. Tamano asked the following question. Is a stratifiable space metrizable if it is κ -metrizable? To answer this question, we show that the class of stratifiable κ -metrizable spaces is much wider than that of metrizable spaces. In fact, we describe two very distinct classes of countable, stratifiable κ -metrizable spaces which are not metrizable. One of them has no nontrivial convergent sequence. The other is bisequential but not a w -space. In addition, we give a characterization of κ -metrizability of countable spaces defined by the Cantor tree and note a topological property of monotonically normal κ -metrizable spaces.

1. INTRODUCTION

We start with some definitions and notation. $\mathbb{N}$ is the set of natural numbers, and ω is the first infinite ordinal. As usual, we identify an ordinal with the set of smaller ordinals. Unexplained notions and terminology are the same as [3]. A space X is called monotonically normal [7] if for each pair (x, U) consisting of a point x in X and its neighborhood U , there exists an open set $H(x, U)$ satisfying the following two conditions: (MN1) $x \in H(x, U) \subset U$; (MN2) if $H(x, U) \cap H(y, V) \neq \emptyset$, then either $x \in V$ or $y \in U$. We call H the monotone operator for X . Monotone normality is a hereditary property. Both generalized ordered spaces and stratifiable spaces are monotonically normal [7]. A space X is called stratifiable [5] if there is a function G which assigns to each $n \in \mathbb{N}$ and each closed set $H \subset X$, an open set $G(n, H)$ containing H such that: (a) $H = \bigcap \overline{G(n, H)}$; (b) if $H \subset K$, then $G(n, H) \subset G(n, K)$. The class of stratifiable spaces is hereditary and countably productive [5, Theorem 5.10]. For example, a monotonically normal σ -space is stratifiable.

According to [15] a κ -metric on a space X is a nonnegative real-valued function $\rho(x, R)$ of two variables—a point x in X and a regular closed subset R in X —satisfying the following requirements:

(K1) $\rho(x, R) = 0$ iff $x \in R$;

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- (K2) if $R \subset R'$, then $\rho(x, R) \geq \rho(x, R')$ for every $x \in X$;
- (K3) $\rho(x, R)$ is continuous with respect to x for every R ;
- (K4) $\rho(x, \bigcup_{\alpha} \overline{R_{\alpha}}) = \inf_{\alpha} \rho(x, R_{\alpha})$ for every increasing transfinite sequence $\{R_{\alpha}\}$.

A space on which there exists a κ -metric is called κ -metrizable. Obviously every metrizable space is κ -metrizable. The product of any collection of κ -metrizable spaces is κ -metrizable [15, Theorem 2]. Not every subspace of a κ -metrizable space is κ -metrizable. But, we can easily see that both dense subspaces and regular closed subspaces of a κ -metrizable space are κ -metrizable.

Dranishnikov investigated κ -metrizable spaces and stratifiable spaces in view of “annihilation” [2].

In [18, Theorem 2.2] Tamano proved that every Lašnev space (i.e., the closed image of a metrizable space) is metrizable if it is κ -metrizable. Motivated by this result, he asked the following problem [18, Problem 2.4]. Is a stratifiable space metrizable if it is κ -metrizable? Later this problem was answered negatively. In [17] the authors gave a nonmetrizable first countable space Z which is stratifiable and κ -metrizable. This example naturally leads us to the following question.

Question. Is every stratifiable κ -metrizable space first countable?

In this paper we would like to answer the above question also in the negative. For convenience, we denote by $\mathcal{SN}$ the class of stratifiable κ -metrizable spaces. We show: (1) there exists a countable space $M \in \mathcal{SN}$ without any nontrivial convergent sequence and (2) there exists a countable space $S \in \mathcal{SN}$ which is bisquential but not a w -space. Each example has only one nonisolated point. This means that the class of stratifiable κ -metrizable spaces is much wider than that of metrizable spaces. In addition, we give a characterization of κ -metrizability of countable size spaces defined by the Cantor tree and note a topological property of monotonically normal κ -metrizable spaces.

Let $S(\omega)$ be the sequential fan with ω converging sequences; in other words, $S(\omega)$ is the quotient space obtained from the topological sum of ω converging sequences by identifying limit points. $S(\omega)$ is not κ -metrizable, because, as described above, every Lašnev space is metrizable if it is κ -metrizable. Let $\mathbb{N} \cup \{p\}$ be the subspace of $\beta\mathbb{N}$, where $p \in \beta\mathbb{N} - \mathbb{N}$. $\mathbb{N} \cup \{p\}$ is not κ -metrizable, because an extremally disconnected κ -metrizable space is discrete [15].

2. COUNTEREXAMPLES

For a set Y we denote by $\mathcal{P}(Y)$ the set of all subsets of Y . We need the following theorem to prove κ -metrizability of our spaces.

Theorem 2.1. *Let $X = Y \cup \{\infty\}$ be a space such that every point of Y is isolated and the specific point ∞ is a G_{δ} -point in X . Then the following are equivalent:*

- (a) X is κ -metrizable.
- (b) *There exists a function $\varphi: \mathcal{P}(Y) \rightarrow [1, \omega]$ with the following conditions:*
 - (1) $F \in \mathcal{P}(Y)$ is closed in X iff $\varphi(F) < \omega$;
 - (2) if $F_1 \subset F_2$, then $\varphi(F_1) \leq \varphi(F_2)$;
 - (3) if $\{F_{\alpha}: \alpha < \tau\}$ is an increasing collection in $\mathcal{P}(Y)$ such that $\varphi(F_{\alpha}) = k < \omega$ for every $\alpha < \tau$, then $\varphi(\bigcup_{\alpha} F_{\alpha}) = k$.

Proof. Assume (a). Let ρ be a κ -metric on X . Put $E_1 = [1, \infty)$ and $E_n = [1/n, 1/n - 1)$, $n \geq 2$. We define a function $\varphi: \mathcal{P}(Y) \rightarrow [1, \omega]$ in the following manner:

$$\varphi(F) = \begin{cases} \omega & \text{if } F \text{ is not closed in } X, \\ k & \text{if } F \text{ is closed in } X \text{ and } \rho(\infty, F) \in E_k. \end{cases}$$

Obviously φ satisfies (1) and (2). We examine (3). Let $\{F_\alpha: \alpha < \tau\}$ be an increasing collection in $\mathcal{P}(Y)$ such that $\varphi(F_\alpha) = k$ for every $\alpha < \tau$. Then $\rho(\infty, \bigcup_\alpha F_\alpha) = \inf_\alpha \rho(\infty, F_\alpha) \in E_k$. This means $\bigcup_\alpha F_\alpha$ is closed in X and $\rho(\infty, \bigcup_\alpha F_\alpha) \in E_k$. Thus $\varphi(\bigcup_\alpha F_\alpha) = k$.

Conversely, assume (b). Since the point ∞ is a G_δ -point, there exists a decreasing sequence $\{G_n: n \in \mathbb{N}\}$ of open neighborhoods of ∞ such that $\{\infty\} = \bigcap G_n$ and $G_1 = X$. Put $D_n = G_n - G_{n+1}$ for every $n \in \mathbb{N}$. Let $\mathcal{R}$ be the set of regular closed subsets of X , and let $\mathcal{R}_1 = \{R \in \mathcal{R}: \infty \in R\}$, $\mathcal{R}_2 = \{R \in \mathcal{R}: \infty \in X - R\}$. For every $x \in X$ and $R \in \mathcal{R}$ we define $\rho(x, R)$ in the following manner:

$$\text{If } R \in \mathcal{R}_1, \text{ then } \rho(x, R) = \begin{cases} 0 & \text{if } x \in R, \\ 1/k & \text{if } x \in D_k - R. \end{cases}$$

$$\text{If } R \in \mathcal{R}_2 \text{ and } \varphi(R) = k, \text{ then } \rho(x, R) = \begin{cases} 0 & \text{if } x \in R, \\ 1/k & \text{if } x \in G_k - R, \\ 1/j & \text{if } x \in D_j - R, j < k. \end{cases}$$

We only need to examine (K4) to show ρ is a κ -metric. Let $\{R_\alpha: \alpha < \tau\}$ be an increasing collection in $\mathcal{R}$, and let $R = \overline{\bigcup_\alpha R_\alpha}$. First, assume that there exists α with $R_\alpha \in \mathcal{R}_1$. Then $R = \bigcup_\alpha R_\alpha \in \mathcal{R}_1$; hence, $\rho(x, R) = \inf_\alpha \rho(x, R_\alpha) = 0$ if $x \in R$, and $\rho(x, R) = \inf_\alpha \rho(x, R_\alpha) = 1/k$ if $x \in D_k - R$. Second, we assume $R_\alpha \in \mathcal{R}_2$ for every $\alpha < \tau$. Put $\varphi(R_\alpha) = k_\alpha$ for $\alpha < \tau$. In case $\{k_\alpha: \alpha < \tau\}$ is infinite, $R = \{\infty\} \cup \bigcup_\alpha R_\alpha \in \mathcal{R}_1$ by $\varphi(\bigcup_\alpha R_\alpha) = \omega$. If $x \in \bigcup_\alpha R_\alpha$, then $\rho(x, R) = \inf_\alpha \rho(x, R_\alpha) = 0$. If $x = \infty$, then $\rho(x, R) = 0$ and $\rho(x, R_\alpha) = 1/k_\alpha$ for $\alpha < \tau$; hence, $\rho(x, R) = \inf_\alpha \rho(x, R_\alpha) = 0$. If $x \in D_k - R$, then $\rho(x, R) = 1/k$ and $\rho(x, R_\alpha) = 1/k$ for some α with $k < k_\alpha$; hence, $\rho(x, R) = \inf_\alpha \rho(x, R_\alpha) = 1/k$. In case $\{k_\alpha: \alpha < \tau\}$ is finite, without loss of generality, we may assume $\varphi(R_\alpha) = k$ for every $\alpha < \tau$. From the condition (3) of φ , $\varphi(\bigcup_\alpha R_\alpha) = k$ and $R = \bigcup_\alpha R_\alpha \in \mathcal{R}_2$. If $x \in R$, then $\rho(x, R) = \inf_\alpha \rho(x, R_\alpha) = 0$. If $x \in G_k - R$, then $\rho(x, R) = \inf_\alpha \rho(x, R_\alpha) = 1/k$. If $x \in D_j - R$ and $j < k$, then $\rho(x, R) = \inf_\alpha \rho(x, R_\alpha) = 1/j$. This completes the proof.

In Theorem 2.1 the condition “ ∞ is a G_δ -point” is essential. Let $X = D(\omega_1) \cup \{\infty\}$ be the one-point compactification of the discrete space $D(\omega_1)$ of cardinality ω_1 . If we set $\varphi(F) = |F|$ for $F \subset D(\omega_1)$, where $\varphi(\emptyset) = 1$, then φ satisfies (1), (2), and (3) in Theorem 2.1. But X is not κ -metrizable (see [15, Theorem 6]).

We recall some convergence properties. A space X is sequential [3] if a set $A \subset X$ is closed iff no sequence in A converges to a point not in A . A space X is Fréchet [3] if for every $A \subset X$ and every $x \in \overline{A}$ there exists a sequence $\{x_n: n \in \mathbb{N}\}$ of points in A converging to x . A space X is strongly Fréchet (i.e., countably bisquential [8]) if for every decreasing sequence $\{A_n: n \in \mathbb{N}\}$ of subsets in X and every $x \in \bigcap \overline{A_n}$, there exists an $x_n \in A_n$ such that

$\{x_n : n \in \mathbb{N}\}$ converges to x . A space X is a w -space [16] if for every sequence $\{A_n : n \in \mathbb{N}\}$ of subsets in X and every $x \in \bigcap \overline{A_n}$, there exists an $x_n \in A_n$ such that $\{x_n : n \in \mathbb{N}\}$ converges to x . A space X is bisequential [8] if every ultrafilter converging to a point $x \in X$ contains a decreasing sequence $\{A_n : n \in \mathbb{N}\}$ converging to x . Obviously,

$$\begin{array}{c} \text{first countable} \rightarrow w\text{-space} \rightarrow \text{strongly Fréchet} \rightarrow \text{Fréchet} \rightarrow \text{sequential.} \\ \searrow \qquad \qquad \nearrow \\ \text{bisequential} \end{array}$$

In [17, Theorem 3.4], it was proved that if a monotonically normal κ -metrizable space is sequential, then it is strongly Fréchet.

Let T be the set of all finite sequences of 0's and 1's, and let C be the Cantor set $\{0, 1\}^\omega$. For $t \in T$ we set $U(t) = \{f \in C : t \subset f\}$, where $t \subset f$ means f is an extension of t . Then $\mathcal{U} = \{U(t) : t \in T\}$ is an open base for the Cantor set consisting of nonempty open-and-closed sets. The base $\mathcal{U}$ satisfies that if $U_1, U_2 \in \mathcal{U}$ and $U_1 \cap U_2 \neq \emptyset$, then $U_1 \subset U_2$ or $U_2 \subset U_1$. We call $\mathcal{U}$ the canonical base for the Cantor set $C = \{0, 1\}^\omega$.

Now we give examples described in the introduction. Recall that $\mathcal{SH}$ is the class of stratifiable κ -metrizable spaces.

Example 2.2. There exists a countable space $M \in \mathcal{SH}$ without any nontrivial convergent sequence.

Proof. Let $\mathcal{U}$ be the canonical base for C . Let $\mathcal{V}$ be the set of all open-and-closed sets in C . Since $\mathcal{V}$ is countable, we may set $\mathcal{V} = \{V_0 = \emptyset\} \cup \{V_n : n \in \mathbb{N}\}$. We denote by $C_p(C, \{0, 1\})$ the space of all continuous functions from C to $\{0, 1\}$ with the topology of pointwise convergence. In other words, $C_p(C, \{0, 1\})$ is the subspace of $\{0, 1\}^C$ consisting of all continuous functions from C to $\{0, 1\}$. We can set $C_p(C, \{0, 1\}) = \{f_0\} \cup \{f_n : n \in \mathbb{N}\}$, where f_0 is the constant function to 0 and f_n is the characteristic function on $V_n \in \mathcal{V}$ in C .

Let L be the space whose underlying set is $\{f_0\} \cup \{f_n : n \in \mathbb{N}\}$ and whose topology is given as follows: every f_n , $n \in \mathbb{N}$, is isolated in L ; and a basic open neighborhood of f_0 is the same as $f_0 \in C_p(C, \{0, 1\})$. Obviously, L is stratifiable. Note that L has nontrivial convergent sequences.

We want to show that L is κ -metrizable. For every $n \in \mathbb{N}$ we fix a finite subset $\mathcal{U}_n$ of $\mathcal{U}$ with $V_n = \bigcup \mathcal{U}_n$. For $A \subset \mathbb{N}$ we set $L(A) = \{f_n : n \in A\}$. Put $Y = \{f_n : n \in \mathbb{N}\}$. We define a function $\varphi : \mathcal{P}(Y) \rightarrow [1, \omega]$ by $\varphi(\emptyset) = 1$ and $\varphi(L(A)) \leq k < \omega$ iff there exists $(U_n) \in \prod_{n \in A} \mathcal{U}_n$ such that if $\mathcal{D}$ is a disjoint family in $\{U_n : n \in A\}$, then $|\mathcal{D}| \leq k$.

We examine the conditions (1), (2), and (3) in Theorem 2.1.

First we prove (1). Assume $L(A)$ is closed in L . Then there exists a finite subset F of C with $F \cap V_n \neq \emptyset$, $n \in A$. For each $n \in A$, choose $U_n \in \mathcal{U}_n$ with $F \cap U_n \neq \emptyset$. If $\mathcal{D}$ is a disjoint family in $\{U_n : n \in A\}$, then $|\mathcal{D}| \leq |F|$. Hence $\varphi(L(A)) \leq |F| < \omega$. Conversely assume $\varphi(L(A)) = k < \omega$. Then there exist $(U_n) \in \prod_{n \in A} \mathcal{U}_n$ and its finite subset $\{U_{n_1}, U_{n_2}, \dots, U_{n_k}\} \subset \{U_n : n \in A\}$ such that $\{U_{n_1}, U_{n_2}, \dots, U_{n_k}\}$ is disjoint, and such that, if $\mathcal{D}$ is a disjoint family in $\{U_n : n \in A\}$, then $|\mathcal{D}| \leq k$. Put $\mathcal{W}_0 = \{U_n : U_n \supset U_{n_j} \text{ for some } j\}$ and $\mathcal{W}_j = \{U_n : U_n \subset U_{n_j}\}$, $1 \leq j \leq k$. Obviously $\{U_n : n \in A\} = \mathcal{W}_0 \cup \mathcal{W}_1 \cup \dots \cup \mathcal{W}_k$ by the “non-Archimedean” property of $\mathcal{U}$. Since any two elements of $\mathcal{W}_j$,

$j \neq 0$, must intersect, each $\mathcal{W}_j$ is linearly ordered by inclusion. Thus we can take $x_j \in \bigcap \mathcal{W}_j$, $j \neq 0$. Put $F = \{x_1, x_2, \dots, x_k\}$. Then for each $n \in A$, $F \cap U_n \neq \emptyset$. Especially for each $n \in A$, $F \cap V_n \neq \emptyset$. This means that $L(A)$ is closed in L .

Condition (2) is easy to see.

We prove (3). Assume $n \in \mathbb{N}$, $A_n \subset \mathbb{N}$, $A_n \subset A_{n+1}$, and $\varphi(L(A_n)) = k < \omega$. Put $A = \bigcup_n A_n$. We show $\varphi(L(A)) \leq k$. For every $n \in A$ we give $\mathcal{U}_n$ the discrete topology. For every $j \in \mathbb{N}$ we set $G_j = \{(U_n) \in \prod_{n \in A} \mathcal{U}_n : \text{if } \mathcal{D} \text{ is a disjoint family in } \{U_n : n \in A_j\}, \text{ then } |\mathcal{D}| \leq k\}$. It is not difficult to see that G_j is a nonempty closed set in $\prod_{n \in A} \mathcal{U}_n$ and $G_j \supset G_{j+1}$. Since $\prod_{n \in A} \mathcal{U}_n$ is compact, we can take $(U_n) \in \bigcap G_j$. This (U_n) is the desired one.

By Theorem 2.1, L is κ -metrizable.

It is known that $C_p(C, \{0, 1\})$ is not Fréchet (see the corollary of Theorem 4 in [4]). Hence we can take a subset E of $C_p(C, \{0, 1\})$ such that $f_0 \in \overline{E} - E$ and E does not contain any convergent sequence to f_0 . Let $\pi: L \rightarrow C_p(C, \{0, 1\})$ be the identity map. Put $M = \pi^{-1}(\{f_0\} \cup E)$. The space M is the desired one. Recall that every regular closed subspace of a κ -metrizable space is κ -metrizable. This completes the proof.

Since $C_p(C, \{0, 1\})$ is dense in $\{0, 1\}^C$, it follows that it is κ -metrizable. But $C_p(C, \{0, 1\})$ is not stratifiable (not even monotonically normal). The fundamental idea of this fact is due to [6] essentially; see also [12, Example 4.13].

We comment on the Cantor tree [10] to give another counterexample. Recall T is the set of all finite sequences of 0's and 1's with the extension order $\subset$, and C is the set of all sequences of 0's and 1's whose domain is ω . In other words, C is the Cantor set $\{0, 1\}^\omega$ and $T = \{f|n : n \in \omega, f \in C\}$, where $f|n$ is the restriction of f to the domain n . We give $T \cup C$ the interval topology. Exactly speaking, every point of T is isolated and a neighborhood of a point $f \in C$ is of the form $\{f\} \cup J$, where J is a cofinite subset of $J(f) = \{f|n : n \in \omega\}$. $T \cup C$ is called the Cantor tree. Since $T \cup C$ is locally compact, it has the one-point compactification $T \cup C \cup \{\infty\}$. We set $S = T \cup \{\infty\}$, the subspace of $T \cup C \cup \{\infty\}$. Note that for $A \subset T$ A is closed in S iff $A \subset J(F)$ for some finite subset F of C , where $J(F) = \bigcup_{f \in F} J(f)$. The space $T \cup C \cup \{\infty\}$ is bisquential [9, 3.4], and hence, S is also bisquential; while $T \cup C \cup \{\infty\}$ is not a w -space ([9, 3.4] or [10, Proposition 2.4]). Thus S is not a w -space.

Example 2.3. There exists a countable space $S \in \mathcal{SH}$ which is bisquential but not a w -space.

Proof. Let S be the space defined above. We have only to see that S is κ -metrizable. For each $t \in T$, let $\chi(t)$ be the characteristic function on $U(t)$ in C . Recall the space L in the proof of Example 2.2. We define a mapping $e: S \rightarrow L$ by $e(\infty) = f_0$ and $e(t) = \chi(t)$. Then e is injective and $e(J(f)) = \{\chi(t) : t \in J(f)\}$. Therefore, e is an embedding into L . Since $e(S)$ is regular closed in L , S is κ -metrizable. This completes the proof.

The space S in Example 2.3 is identical with the space defined in [13, Example].

3. κ -METRIZABILITY OF $S(K)$

For $K \subset C$, let $T \cup K$ be the space with the interval topology and $T \cup K \cup \{\infty\}$ be the one-point compactification of $T \cup K$. We set $S(K) = T \cup \{\infty\}$, the subspace of $T \cup K \cup \{\infty\}$. Note that for $A \subset T$ A is closed in $S(K)$ iff $|A - J(F)| < \omega$ for some finite subset F of K . The space S is identical with $S(C)$. In view of Example 2.3, one naturally asks the question “is $S(K)$ κ -metrizable for any $K \subset C$?”. In this section we answer this question in the negative. In fact, we show that $S(K)$ is κ -metrizable iff K is an F_σ -set in C . Note that $S(K)$ is metrizable iff K is countable.

Lemma 3.1 [11, Lemma 1]. *Let H be a monotone operator on a monotonically normal space X , let $Y \subset X$, and let $\{U_y : y \in Y\}$ be an open family in X such that $y \in U_y$ for every $y \in Y$. Then*

$$\overline{\bigcup_{y \in Y} H(y, U_y)} \subset \overline{Y} \cup \bigcup_{y \in Y} U_y.$$

Lemma 3.2. *Let $X = Y \cup \{\infty\}$ be a space such that every point of Y is isolated in X . If X is a subspace of a monotonically normal κ -metrizable space Z , then X is κ -metrizable.*

Proof. We may assume that the specific point ∞ is not isolated in X . Let H be a monotone operator on Z and ρ a κ -metric on Z . For every $y \in Y$ we pick an open set U_y in Z with $U_y \cap X = \{y\}$. Put $V_y = H(y, U_y)$. Let $\mathcal{R}$ be the set of regular closed subsets of X . For $x \in X$ and $R \in \mathcal{R}$ we set $\delta(x, R) = \rho(x, \overline{\bigcup\{V_y : y \in Y \cap R\}})$. We see that δ is a κ -metric on X . Note that for $x \in X$, $A \subset Y$, and $R \in \mathcal{R}$, $x \in \overline{\bigcup_{y \in A} V_y}$ iff $x \in \overline{A}$ by virtue of Lemma 3.1 and $x \in R$ iff $x \in \overline{Y \cap R}$. This fact implies that δ satisfies (K1). It is easy to see that (K2) and (K3) hold. Last of all we examine (K4). Let $\{R_\alpha : \alpha < \tau\}$ be an increasing collection in $\mathcal{R}$. Then

$$\begin{aligned} \delta\left(x, \overline{\bigcup_{\alpha} R_{\alpha}}\right) &= \rho\left(x, \overline{\bigcup\left\{V_y : y \in Y \cap \left(\bigcup_{\alpha} R_{\alpha}\right)\right\}}\right) \\ &\quad \left(\text{where } \overline{\bigcup_{\alpha} R_{\alpha}} \text{ is the closure in } X\right) \\ &= \rho\left(x, \overline{\bigcup_{\alpha} C_{\alpha}}\right) \quad \left(\text{where } C_{\alpha} = \overline{\bigcup\{V_y : y \in Y \cap R_{\alpha}\}}\right) \\ &= \inf_{\alpha} \rho(x, C_{\alpha}) = \inf_{\alpha} \delta(x, R_{\alpha}). \end{aligned}$$

This completes the proof.

Lemma 3.3. *Let $K = \bigcup\{K_n : n \in \mathbb{N}\}$ be a subset of C . If each $S(K_n)$ is κ -metrizable, then $S(K)$ is κ -metrizable.*

Proof. Note that $P = \prod_{n \in \mathbb{N}} S(K_n)$ is stratifiable and κ -metrizable. Put $D = \{(\infty, \infty, \dots)\} \cup \{(t, t, \dots) : t \in T\} \subset P$. Every $(t, t, \dots)$ is isolated in D . Hence D is κ -metrizable by Lemma 3.2. We define a map $\pi : D \rightarrow S(K)$ by $\pi((\infty, \infty, \dots)) = \infty$ and $\pi((t, t, \dots)) = t$. If $f \in K_n$, then $\pi^{\leftarrow}(J(f)) = (\prod_{m < n} S(K_m) \times J(f) \times \prod_{m > n} S(K_m)) \cap D$ is closed in D . This implies that π

is a homeomorphism. Consequently $S(K)$ is κ -metrizable. This completes the proof.

Theorem 3.4. *Let K be a subset of C . Then $S(K)$ is κ -metrizable iff K is an F_σ -set in C .*

Proof. Assume $S(K)$ is κ -metrizable. Let φ be a function as in Theorem 2.1. We set $K_n = \{f \in C : \varphi(J(f)) \leq n\}$ for every $n \in \mathbb{N}$. Obviously $K = \bigcup_{n \in \mathbb{N}} K_n$. We show that each K_n is closed in C . Assume $g \in \overline{K_n}$. Then for every $m \in \omega$ there exists an $f_m \in K_n$ with $g|_m \subset f_m$. If we set $J_m = \{g|_k : k \leq m\}$, then $\varphi(J_m) \leq \varphi(J(f_m)) \leq n$. Hence $\varphi(J(g)) = \varphi(\bigcup_{m \in \omega} J_m) \leq n$ by the condition (3) of φ . Thus $g \in K_n$.

Conversely, assume that K is an F_σ -set in C . By Lemma 3.3 we may assume that K is closed in C . We choose a countable subset $\{t_n : n \in \omega\} \subset T$ with $C - K = \bigcup_{n \in \omega} U(t_n)$. Put $E = \{t \in T : t_n \subset t \text{ for some } n \in \omega\}$. Note $E \cap J(K) = \emptyset$. For $A \subset T$ we set $\varphi_1(A) = \sup\{|B| : B \subset A \text{ and } B \text{ is an antichain}\}$ and $\varphi_2(A) = |A \cap E|$, where a subset B of T is called an antichain if for any two distinct $s, t \in B$ neither $s \subset t$ nor $t \subset s$ holds. Put $\varphi(A) = \max\{\varphi_1(A), \varphi_2(A)\}$, where $\varphi(\emptyset) = 1$. We see that φ satisfies (1), (2), and (3) in Theorem 2.1. Assume A is closed in $S(K)$. Then $|A - J(F)| < \omega$ for some finite subset F of K . This implies $\varphi_1(A) < \omega$ and $\varphi_2(A) < \omega$. Conversely, assume $\varphi(A) < \omega$. Let $\varphi_1(A) = k$ and $\{s_1, s_2, \dots, s_k\}$ be an antichain subset of A . Since every $t \in A$ is compatible with some s_i , $A = \{t \in A : t \subset s_i \text{ for some } i\} \cup A_1 \cup \dots \cup A_k$, where $A_i = \{t \in A : s_i \subset t\}$. Each A_i must be linearly ordered. If A_i is infinite and $A_i \subset J(g)$, then $g \in K$. Because, if $g \in C - K$, then $g \in U(t_j)$ for some $j \in \omega$, which implies $|A_i \cap E| = \omega$, contradicting $\varphi_2(A) < \omega$. Thus $|A - J(F)| < \omega$ for some finite subset F of K . Thus A is closed in $S(K)$. Therefore, φ satisfies (1). Other conditions are easy to see. By Theorem 2.1 $S(K)$ is κ -metrizable. This completes the proof.

Corollary 3.5. *Let K be the set of irrationals in C . Then $S(K)$ is not κ -metrizable.*

4. FAN TIGHTNESS OF κ -METRIZABLE SPACES

In this section we note a topological property of monotonically normal κ -metrizable spaces.

Example 2.2 shows that a stratifiable κ -metrizable space does not always have convergence properties. However, we prove that X^ω has countable fan tightness provided that X is monotonically normal and κ -metrizable. A space X is said to have countable fan tightness [1] if for each $x \in X$ and each countable system $\{A_n : n \in \mathbb{N}\}$ of subsets of X such that $x \in \bigcap \overline{A_n}$ there exist finite sets B_n of A_n such that $x \in \bigcup \overline{B_n}$. If each B_n is replaced with a singleton subset in the above definition, then X is said to have countable strong fan tightness [14].

Obviously a space with countable fan tightness has countable tightness. The sequential fan $S(\omega)$ has countable tightness but does not have countable fan tightness.

Lemma 4.1 [15, Lemma 4]. *Let $\{R_\alpha : \alpha \in A\}$ be a family of regular closed subsets of a κ -metric space (X, ρ) . Then for any $\varepsilon > 0$ and $x \in X$ there exist $\alpha_1, \dots, \alpha_n \in A$ such that $\rho(x, \bigcup_{i=1}^n R_{\alpha_i}) < \rho(x, \bigcup_{\alpha \in A} R_\alpha) + \varepsilon$.*

Proposition 4.2. *If X is monotonically normal and κ -metrizable, then X^ω has countable fan tightness.*

Proof. Let H be a monotone operator on X and ρ be a κ -metric on X . Note that X^ω has countable fan tightness if each finite product of X has countable fan tightness. First we show X has countable fan tightness. Assume $A_n \subset X$ and $x \in \bigcap \overline{A_n}$, $n \in \mathbb{N}$. We may assume $x \in X - \bigcup A_n$. For every $y \in \bigcup A_n$ we set $U_y = H(y, X - \{x\})$. By Lemma 4.1, there exists a finite set B_n of A_n such that $\rho(x, \bigcup_{y \in B_n} \overline{U_y}) < 1/n$. Put $B = \bigcup B_n$. Then $\rho(x, \bigcup_{y \in B} \overline{U_y}) = 0$. This means $x \in \bigcup_{y \in B} \overline{U_y}$. By Lemma 3.1, $\bigcup_{y \in B} \overline{U_y} \subset \overline{B} \cup X - \{x\}$. Thus $x \in \overline{B}$. Second, we see X^2 has countable fan tightness. Assume $A_n \subset X^2$ and $(x, y) \in \bigcap \overline{A_n}$, $n \in \mathbb{N}$. We may assume $(u, v) \in X^2 - \{x\} \times X \cup X \times \{y\}$ if $(u, v) \in \bigcup A_n$, because X has countable fan tightness. For every $(u, v) \in \bigcup A_n$ we set $U(u, v) = H(u, X - \{x\}) \times H(v, X - \{y\})$. By the same idea as the case of X (recall X^2 is κ -metrizable), there exist finite sets B_n of A_n such that $(x, y) \in \bigcup \{U(u, v) : (u, v) \in B\}$, where $B = \bigcup B_n$. Also, by Lemma 3.1, $\bigcup \{U(u, v) : (u, v) \in B\} \subset \overline{B} \cup X^2 - \{(x, y)\}$. Thus $(x, y) \in \overline{B}$. By induction, each finite product of X has countable fan tightness. This completes the proof.

Remark. A stratifiable κ -metrizable space does not always have countable strong fan tightness. Since $C_p(C, \{0, 1\})$ does not have countable strong fan tightness [14], there exists a sequence $\{A_n : n \in \mathbb{N}\}$ of subsets in $C_p(C, \{0, 1\})$ such that $f_0 \in \bigcap \overline{A_n}$; but for any $h_n \in A_n$ $f_0 \in C_p(C, \{0, 1\}) - \{h_n : n \in \mathbb{N}\}$. This implies that L in Example 2.2 does not have countable strong fan tightness.

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