A CURVE FOR WHICH COLEMAN'S EFFECTIVE CHABAUTY BOUND IS SHARP

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ABSTRACT. We show that Coleman's effective Chabauty bound is sharp for the curve $C : y^2 = x(x-1)(x-2)(x-5)(x-6)$ defined over $\mathbb{Q}$, by considering its reduction mod 7. We also show that the Jacobian of C is absolutely simple.

Let C be the curve

$$(1) \quad y^2 = x(x-1)(x-2)(x-5)(x-6)$$

defined over $\mathbb{Q}$ and $J(C)$ its Jacobian. Recently Gordon and the author computed that $J(C)(\mathbb{Q}) \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^4$ [GG].

Proposition 1. *Let P_∞ denote the point at infinity on C . Then*

$$C(\mathbb{Q}) = \{P_\infty, (0, 0), (1, 0), (2, 0), (5, 0), (6, 0), \\ (3, 6), (3, -6), (10, 120), (10, -120)\}.$$

Proof. The genus of C is 2, which is greater than the Mordell-Weil rank of $J(C)(\mathbb{Q})$, so Chabauty's Theorem [Ch] shows that $\#C(\mathbb{Q})$ is finite (of course so does Faltings's Theorem [F]). For any q a power of a prime, we let $\mathbb{F}_q$ be the field with q elements. The proof of Corollary 4.6 of [Co] shows that if X is a curve of genus $g \geq 2$ over $\mathbb{Q}$, if the Mordell-Weil rank of the Jacobian of X over $\mathbb{Q}$ is less than g , if $p > 2g$ is a prime of good reduction for X , and if $\tilde{X}$ is the reduction of X mod p , then $\#X(\mathbb{Q}) \leq \#\tilde{X}(\mathbb{F}_p) + 2g - 2$. The proposition follows once we note that C has good reduction at $p = 7$, that $\#\tilde{C}(\mathbb{F}_7) = 8$, and that the stated points lie on the curve.

So far as I know, C is the first example of a curve (whose Jacobian has Mordell-Weil rank > 0) whose rational points were determined by Coleman's bound. (See also [M].) Of course if C covered an elliptic curve over $\mathbb{Q}$ of rank 0, then $C(\mathbb{Q})$ could also be determined via the cover. We now show that C covers no elliptic curves.

Proposition 2. *$J(C)$ is absolutely simple.*

Proof. The following argument was worked out jointly with Jaap Top. If $J(C)$ were not absolutely simple, then there would be a pair of elliptic curves E_1 and

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E_2 and an isogeny $\phi: J(C) \rightarrow E_1 \times E_2$, where E_1 , E_2 , and ϕ are all defined over some number field K . We will derive a contradiction by comparing the L -function of C with those of E_1 and E_2 .

Note that by (1), C has good reduction at the prime $p = 11$. Let $\tilde{C}$ denote the reduced curve. It is shown in [T] that if $N_5 = \#\tilde{C}(\mathbb{F}_{11^5})$, then the L -function of $\tilde{C}$ over $\mathbb{F}_{11}$ is $L(\tilde{C}/\mathbb{F}_{11}, t) = 1 - at + bt^2 - 11at^3 + 121t^4$, where $a = 11 + 1 - N_1$ and $b = (1/2)(N_2 + N_1^2) - (1 + 11)N_1 + 11$. A calculation shows that $\#\tilde{C}(\mathbb{F}_{11}) = 16$ and $\#\tilde{C}(\mathbb{F}_{121}) = 118$, so $L(\tilde{C}/\mathbb{F}_{11}, t) = 1 + 4t + 6t^2 + 44t^3 + 121t^4$.

By the functional equation $L(\tilde{C}/\mathbb{F}_{11}, t) = 11^2 t^4 L(\tilde{C}/\mathbb{F}_{11}, 1/11t)$, the L -function factors over its splitting field as $(1 - \alpha t)(1 - (11/\alpha)t)(1 - \beta t) \times (1 - (11/\beta)t)$, for some α, β . So if we set $(1 - \alpha t)(1 - (11/\alpha)t) = 1 + ut + 11t^2$, $(1 - \beta t)(1 - (11/\beta)t) = 1 + vt + 11t^2$, we can take $u = 2 + 2\sqrt{5}$, $v = 2 - 2\sqrt{5}$. Therefore if $\zeta_5 = e^{2\pi i/5}$, then $\zeta_5 + \zeta_5^{-1} = (\sqrt{5} - 1)/2$, and solving the quadratics we get

$$(2) \quad \begin{aligned} \alpha &= -2\zeta_5^2 - 2\zeta_5^3 \pm (\zeta_5 - \zeta_5^2 + \zeta_5^3 - \zeta_5^4), \\ \beta &= -2\zeta_5 - 2\zeta_5^4 \pm (\zeta_5 + \zeta_5^2 - \zeta_5^3 - \zeta_5^4). \end{aligned}$$

Let $\mathcal{P}$ be a prime of K above 11, and suppose that the absolute norm of $\mathcal{P}$ is 11^f . Then $\mathcal{P}$ is a prime of good reduction for C , and so also for J and E_1 and E_2 .

Let ℓ be a prime not dividing 11. Recall that for any s , the Frobenius Fr_{11^s} on $J(\tilde{C})$ over $\mathbb{F}_{11^s}$ induces an endomorphism $\text{Fr}_{11^s}^*$ on the Tate module $T_\ell(J(\tilde{C}))$, and that $L(\tilde{C}/\mathbb{F}_{11^s}, t) = \det(1 - \text{Fr}_{11^s}^* t | T_\ell(J(\tilde{C})))$. For any prime ℓ not dividing the degree of ϕ , we get an isomorphism $T_\ell(J(C)) \cong T_\ell(E_1) \times T_\ell(E_2)$ over K , and so using the functional equations of $L(\tilde{E}_i/\mathbb{F}_{11^f}, t)$ for $i = 1, 2$, we have that

$$(3) \quad \begin{aligned} L(\tilde{C}/\mathbb{F}_{11^f}, t) &= L(\tilde{E}_1/\mathbb{F}_{11^f}, t) L(\tilde{E}_2/\mathbb{F}_{11^f}, t) \\ &= (1 - a_1 t + 11^f t^2)(1 - a_2 t + 11^f t^2), \end{aligned}$$

for some integers a_1 and a_2 . But the Frobenius on $J(C)$ over $\mathbb{F}_{11^f}$ is $(\text{Fr}_{11})^f$, so

$$(4) \quad L(\tilde{C}/\mathbb{F}_{11^f}, t) = (1 - \alpha^f t)(1 - (11/\alpha)^f t)(1 - \beta^f t)(1 - (11/\beta)^f t).$$

Renumbering E_1 and E_2 if necessary, from (3) and (4) we can set

$$(1 - \alpha^f t)(1 - (11/\alpha)^f t) = (1 - a_1 t + 11^f t^2),$$

which means that $[\mathbb{Q}(\alpha^f) : \mathbb{Q}] = 1$ or 2 . Since $\alpha \in \mathbb{Q}(\zeta_5)$, which has a unique quadratic subfield, we can assume $\alpha^f \in \mathbb{Q}(\sqrt{5})$. If we let σ denote complex conjugation, then $\alpha^f = \sigma(\alpha^f) = (\sigma(\alpha))^f$; so $(\alpha/\sigma(\alpha))^f = 1$. Therefore $(\alpha/\sigma(\alpha))$ is a root of unity in $\mathbb{Q}(\zeta_5)$, and hence a 10^{th} -root of unity. Therefore $\alpha^{10} \in \mathbb{Q}(\sqrt{5})$. But a calculation with (2) shows this is not the case, so $J(C)$ is absolutely simple.

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