ELLIPTIC GENERA OF LEVEL N AND JACOBI POLYNOMIALS

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ABSTRACT. In this work, we study Hirzebruch's level N elliptic genera and show that the image of the complex projective spaces under the level 3 genus can be realized very compactly in terms of Jacobi polynomials. To obtain these results we examine a differential equation which the level 3 logarithm satisfies.

1. INTRODUCTION

The work in this paper is concerned with the study of the Hirzebruch level N elliptic genus φ_N . This area of mathematics has thus far proven very rich. For example, the complete description of φ_2 led to the discovery of elliptic cohomology theory (cf. [L1, L2]). Thus the understanding of the level N elliptic genus and its formal group law may in turn provide insight into the existence of level N ($N \geq 3$) elliptic cohomology theories. Knowledge of these theories is not only of interest to topologists, but also to number theorists and physicists (cf. [Br, C-C, W]).

Our main objective in this paper is to prove the following:

Theorem 4.1. *The image of the complex projective spaces under the level 3 genus can be realized as "homogeneous" Jacobi polynomials.*

In order to calculate the images of the complex projective spaces we use the level N logarithm. If we expand this logarithm as a power series in a neighborhood of zero it is well known that the coefficients are exactly $\varphi_N(CP^n)/(n+1)$ (cf. [H2]). Thus knowing the coefficients of the level N logarithm is essential in solving the problem. In [L1] and [L2] Landweber shows that for level 2 these coefficients can be written in terms of Legendre polynomials, and concludes that $\text{Im } \varphi_2 \subset Z[1/2][\delta, \varepsilon]$, where δ and ε are modular forms of weights two and four, respectively. For the level 3 case the problem is reduced to solving a cubic polynomial equation and noticing (after some algebraic manipulation) that the coefficients of the level 3 logarithm can be written out in terms of Jacobi polynomials (note that Legendre polynomials are a special case of Jacobi polynomials). These results naturally lead us to conjecture that for higher levels the coefficients of all the level N logarithms can be expressed in terms of Jacobi polynomials.

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For level 2 Landweber [L1, L2] shows that the existence of elliptic cohomology depends on certain congruences for Legendre polynomials modulo powers of all odd primes. These congruences imply that after inverting ε or $\delta^2 - \varepsilon$ the hypothesis of the exact functor theorem is satisfied, thereby leading to an elliptic cohomology theory. For the level 3 theory it should also be the case that the existence of a level 3 elliptic cohomology theory is dependent on certain congruences for Jacobi polynomials modulo powers of primes.

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2. LEVEL N ELLIPTIC GENERA

Let L be a lattice in the complex plane C generated by ω_1 and ω_2 with $\text{Im}(\omega_2/\omega_1) > 0$. After multiplying L by a nonzero constant we can assume that $L = 2\pi i(Z\tau + Z)$, i.e., $\omega_1 = 2\pi i$ and $\omega_2 = 2\pi i\tau$, where τ is in the upper half plane $\mathcal{H}$. Let

$$z = 2\pi i \left(\frac{k}{N}\tau + \frac{l}{N} \right) \neq 0$$

be a fixed N -division point of the lattice with $0 \leq k, l < N$. Furthermore, we assume, in case $N > 2$ and $N = 2m$, that $(k, l) \neq (m, 0), (0, m)$ or (m, m) , so that z is not of order 2 in C/L .

We now define the complex valued function f_z by

$$(2.1) \quad f_z(x) = \frac{\sigma(x)\sigma(z)}{\sigma(z-x)} e^{-k_0 x},$$

where $\sigma(x)$ is the Weierstrass sigma function for the lattice L and $k_0 = \frac{2k}{N}\eta_2 + \frac{2l}{N}\eta_1$. Here $\eta_2 = \zeta(\pi i\tau)$ and $\eta_1 = \zeta(\pi i)$, where $\zeta(z)$ is the Weierstrass zeta function (cf. [Ch]).

It can be shown that

$$(2.2) \quad f_z(x + 2\pi i\tau) = e^{-2\pi i l/N} f_z(x)$$

and

$$(2.3) \quad f_z(x + 2\pi i) = e^{2\pi i k/N} f_z(x)$$

by using properties of the sigma function (cf. [Ch, pp. 51–55]) and the Legendre relation $2\tau\eta_1 - 2\eta_2 = 1$ (cf. [Ch, Theorem 2, p. 50]).

Thus (2.2) and (2.3) imply that f_z is elliptic on a sublattice of L . We see that $F_z(x) = f_z(x)^N$ is an elliptic function on L with a zero of order N at the origin and a pole of order N at z , i.e., on C/L the divisor of F_z is

$$(2.4) \quad \text{div } F_z(x) = N(0) - N(z).$$

We note further that $f_z(x) = x + \cdots$ in a neighborhood of zero.

Example 2.0.1. For $N = 2$ we have that $(k, l) = (1, 0), (0, 1),$ or $(1, 1)$, and thus $k_0 = k\eta_2 + l\eta_1$.

From this point on we will omit the subscript z .

Definition 2.1. The *Hirzebruch level N elliptic genus* is the ring homomorphism

$$\varphi_N: \Omega_*^U \rightarrow C$$

defined by

$$\varphi_N(M^{2k}) = \prod_{i=1}^k \frac{x_i}{f(x_i)} [M^{2k}].$$

Here the x_i are given by applying the splitting principle to the total Chern class, $c(M^{2k}) = \prod_{i=1}^k (1 + x_i)$, and $[M^{2k}]$ is the fundamental homology class of the U -manifold M^{2k} (cf. [H3]). Note that since f is dependent on $\tau \in \mathcal{H}$ as a parameter, φ_N also depends on τ .

Closely associated to any genus with values in a Q -algebra is its logarithm (cf. [R, p. 358]). For the level N elliptic genus we have the logarithm defined by

$$(2.5) \quad g(x) = \sum_{n \geq 0} \frac{\varphi_N(CP^n)}{n+1} x^{n+1}.$$

Thus,

$$(2.6) \quad g'(x) = \sum_{n \geq 0} \varphi_N(CP^n) x^n.$$

By using a residue argument one can show $f(x) = g^{-1}(x)$ in a neighborhood of the origin (cf. [H2]). A similar residue argument yields:

Lemma 2.1 ([H3]). $\varphi_N(CP^m) = 0$ if $m+1$ is divisible by N .

Lemma 2.2. If z is our fixed N -division point then for each $n \geq 1$

$$(2.7) \quad \wp^{(n)}(z) = (-1)^n (n+1)! N^{n+2} G_{n+2}^{(k,l)}(\tau),$$

where $G_{n+2}^{(k,l)}(\tau)$ is an Eisenstein series of level N and weight $n+2$ and $\wp^{(n)}(u)$ denotes the n th derivative of the Weierstrass $\wp$ -function (cf. [Ko, pp. 131–134]).

Proof. For any $n \geq 1$ we have

$$\begin{aligned} \wp^{(n)}(z) &= (-1)^n (n+1)! \sum_{\omega \in L} \frac{1}{(z - \omega)^{n+2}} \\ &= (-1)^n (n+1)! \sum_{(\alpha, \beta) \in \mathbb{Z} \times \mathbb{Z}} \frac{1}{\left(\frac{2\pi i k \tau}{N} + \frac{2\pi i l}{N} - 2\pi i \alpha \tau - 2\pi i \beta\right)^{n+2}}, \end{aligned}$$

where we are writing $\omega = 2\pi i \alpha \tau + 2\pi i \beta$. If we factor out a power of N from the denominator we have, by the definition [Ko, p. 131] of the Eisenstein series,

$$\begin{aligned} \wp^{(n)}(z) &= (-1)^n (n+1)! N^{n+2} \sum_{(\alpha, \beta) \in \mathbb{Z} \times \mathbb{Z}} \frac{1}{(2\pi i (k - N\alpha)\tau + 2\pi i (l - N\beta))^{n+2}} \\ &= (-1)^n (n+1)! N^{n+2} G_{n+2}^{(k,l)}(\tau). \quad \square \end{aligned}$$

Proposition 2.3. As a formal power series

$$\frac{x f'(x)}{f(x)} = 1 + \sum_{r \geq 1} e_r x^r,$$

where e_r is a modular form of weight r for the group

$$\{A \in SL_2(\mathbb{Z}) : (k, l)A \equiv (k, l) \pmod{N}\}.$$

Proof. By taking logarithmic derivatives one finds immediately that

$$(2.8) \quad \frac{xf'(x)}{f(x)} = x\zeta(x) + x\zeta(z-x) - xk_0.$$

It is known that $x\zeta(x) = 1 - \sum_{n \geq 2} G_{2n}(\tau)x^{2n}$, where $G_{2n}(\tau)$ is the classical Eisenstein series of weight $2n$. It is also known that each $G_{2n}(\tau)$ is a modular form of weight $2n$ for $SL_2(\mathbb{Z})$ (cf. [Ko, pp. 109–110]).

Likewise, from the Taylor expansion of $\wp(u-x)$ about $x=0$ one finds that

$$(2.9) \quad x\zeta(u-x) = x\zeta(u) + \sum_{n \geq 2} \frac{(-1)^n}{(n-1)!} \wp^{(n-2)}(u)x^n$$

in a neighborhood of zero since $\zeta'(x) = -\wp(x)$.

Lemma 2.2 immediately implies that

$$\frac{xf'(x)}{f(x)} = 1 - \sum_{n \geq 2} G_{2n}(\tau)x^{2n} + \sum_{n > 2} N^n G_n^{(k,l)}(\tau)x^n + x(\zeta(z) - k_0) + \wp(z)x^2$$

and thus,

$$(2.10) \quad \begin{cases} e_1 = \zeta(z) - k_0, \\ e_2 = \wp(z), \\ e_{2r} = -G_{2r}(\tau) + N^{2r} G_{2r}^{(k,l)}(\tau), & r \geq 2, \\ e_{2r+1} = N^{2r+1} G_{2r+1}^{(k,l)}(\tau), & r \geq 1. \end{cases}$$

This proves that e_r is a modular form of weight r for the indicated groups for all $r \geq 1$. $\square$

Corollary 2.4. For $N > 2$, the modular form e_3 does not vanish on the upper half plane.

Proof. We have seen that $\wp'(z) = -2N^3 G_3^{(k,l)}(\tau) = -2e_3$. Since $\wp'(z)$ only vanishes at the three 2-division points and $2z \notin L$ (by our assumption on z) we conclude that e_3 cannot vanish on the upper half plane. $\square$

We note that $f(x) = x \exp(\sum_{r \geq 1} \frac{e_r}{r} x^r)$, as a consequence of the expression given by Proposition 2.3.

Proposition 2.5 ([H3, p. 39]). We have

$$f(x) = e^{kx/N} \Phi(x) \Phi(-z) / \Phi(x-z),$$

where

$$\Phi(x) = \frac{(1 - e^{-x}) \prod_{n \geq 1} (1 - q^n e^x)(1 - q^n e^{-x})}{\prod_{n \geq 1} (1 - q^n)^2}$$

and $q = e^{2\pi i \tau}$.

By using the additive expansion of the Weierstrass zeta function in terms of q we obtain

$$(2.11) \quad \frac{x f'(x)}{f(x)} = \sum_{n \geq 1} \frac{x q^n e^{-x}}{1 - q^n e^{-x}} - \sum_{n \geq 1} \frac{x q^n e^x}{1 - q^n e^x} + \frac{x e^{-x}}{1 - e^{-x}} + \sum_{n \geq 1} \frac{x q^n e^{x-z}}{1 - q^n e^{x-z}} \\ - \sum_{n \geq 1} \frac{x q^n e^{-x+z}}{1 - q^n e^{-x+z}} - \frac{x e^{-x+z}}{1 - e^{-x+z}} + x \frac{k}{N}.$$

Note that if we let $k = 0$ in (2.11) then the k/N term disappears and e^z becomes an N th root of unity.

We can now determine the constant terms of the q -expansions of the e_r by examining (2.11) at the cusp $\tau = i\infty$, i.e., at $q = 0$. For example, if $N = 3$ we find the constant terms for e_1 , e_2 , and e_3 to be $(2k-3)/6$, $1/12$, and 0 , respectively, when $K \neq 0$. If $k = 0$ and $l = 1$ we see that the constant terms are $-i\sqrt{3}/6$, $-1/4$, and $i\sqrt{3}/18$, respectively. For the case $k = 0$, $l = 2$ we just get the conjugates of the $k = 0$, $l = 1$ case.

3. THE DIFFERENTIAL EQUATION

In this section we study a differential equation which proves very valuable in the following section.

Proposition 3.1 ([H2, §7.2]). *The function $f(x)$ satisfies a differential equation of the following form:*

$$(3.1) \quad \left(\frac{f'(x)}{f(x)} \right)^N + a_1 \left(\frac{f'(x)}{f(x)} \right)^{N-1} + \cdots + a_{N-1} \frac{f'(x)}{f(x)} + a_N = \frac{1}{f(x)^N} + a_{2N} f(x)^N.$$

Note that by using a standard argument for elliptic functions we obtain $f(x)f(z-x) = a_{2N}^{-1/N}$.

Proposition 3.2 ([H3]). *The coefficient a_{N-1} appearing in the differential equation (3.1) vanishes: $a_{N-1} = 0$.*

Example 3.0.2. For $N = 2$ the differential equation has the form

$$(3.2) \quad \left(\frac{f'(x)}{f(x)} \right)^2 + a_2 = \frac{1}{f(x)^2} + a_4 f(x)^2.$$

Rewriting (3.2) as $f'(x)^2 = 1 - a_2 f(x)^2 + a_4 f(x)^4$ we see that (in the notation used in [L1, L2, O]) $a_2 = 2\delta$ and $a_4 = \varepsilon$.

If we multiply both sides of equation (3.2) by x^2 and make the substitutions $\frac{x f'(x)}{f(x)} = 1 + \sum_{r \geq 1} e_r x^r$ and $f(x) = x \exp(\sum_{r \geq 1} \frac{e_r}{r} x^r)$ we see that $a_2 = -3e_2$ and $a_4 = \frac{1}{2}e_2^2 + \frac{5}{2}e_4$. Thus the coefficients a_2 and a_4 are modular forms of weights 2 and 4 respectively.

Likewise, for level 3 we get the differential equation

$$(3.3) \quad \left(\frac{f'(x)}{f(x)} \right)^3 + a_1 \left(\frac{f'(x)}{f(x)} \right)^2 + a_3 = \frac{1}{f(x)^3} + a_6 f(x)^3.$$

By using the same substitutions as for the level 2 case we obtain $a_1 = -6e_1$, $e_2 = 3e_1^2$ (since $a_2 = 0$), $a_3 = 32e_1^3 - 4e_3$, and $a_6 = 4e_3^2$. Thus, the coefficients are modular forms of weights one, three, and six respectively.

It is known that $a_1, a_2, \dots, a_{N-2}, a_N, a_{2N}$ are modular forms of weights $1, \dots, N-2, N$, and $2N$, respectively (cf. [H3]).

4. LEVEL 3 ELLIPTIC GENERA

In this section we explore the elliptic genus when N is 3 and show that the images of the complex projective spaces, under the genus, can be written very nicely in terms of Jacobi polynomials. The level 2 case has been thoroughly examined by Landweber, Stong, Ochanine, and Ravenel (cf. [L1, L2, O]). We know that the coefficients of the level N logarithm, expanded in a neighborhood of zero, depend on the images of the complex projective spaces. In fact it is enough to know the coefficients of the derivative of the logarithm (2.6). To understand these coefficients we will use the differential equation (3.1) and make the substitution $x = g(u)$, where g is the level N logarithm. By doing this we obtain a new differential equation which g satisfies, i.e.,

$$(4.1) \quad g'(u)^N (1 - a_N u^N + a_{2N} u^{2N}) - a_{N-2} (u g'(u))^{N-2} - \dots - a_1 u g'(u) - 1 = 0.$$

Theorem 4.1. *For $N = 3$ the coefficients of $g'(u)$ can be realized as "homogeneous" Jacobi polynomials, i.e.,*

$$\varphi_3(CP^n) = \begin{cases} (\sqrt{a_6})^k P_k^{(0, -1/3)}(a_3/2\sqrt{a_6}), & n = 3k, \\ \frac{a_1}{3} (\sqrt{a_6})^k P_k^{(0, 1/3)}(a_3/2\sqrt{a_6}), & n = 3k + 1, \\ 0, & n = 3k + 2. \end{cases}$$

Proof. For the level 3 situation the differential equation (4.1) is written as

$$(4.2) \quad g'(u)^3 (1 - a_3 u^3 + a_6 u^6) - a_1 u g'(u) - 1 = 0.$$

If we let $H(u) = 1 - a_3 u^3 + a_6 u^6$ we can then rewrite (4.2), after dividing by $H(u)$, as

$$(4.3) \quad g'(u)^3 - \frac{a_1 u}{H(u)} g'(u) - \frac{1}{H(u)} = 0.$$

Equation (4.3) can now be viewed as a cubic polynomial equation with coefficients $-a_1 u/H(u)$ and $-1/H(u)$.

After solving this cubic polynomial equation and using formal algebra we obtain

$$(4.4) \quad g'(u) = H(u)^{-1/2} \left\{ \left(\frac{1 + 2e_3 u^3 + H(u)^{1/2}}{2} \right)^{1/3} + \left(\frac{-1 - 2e_3 u^3 + H(u)^{1/2}}{2} \right)^{1/3} \right\}.$$

By a fairly straightforward algebraic argument one can show

$$\left(\frac{-1 - 2e_3 u^3 + H(u)^{1/2}}{2} \right)^{1/3} = -2e_1 u \left(\frac{1 + 2e_3 u^3 + H(u)^{1/2}}{2} \right)^{-1/3}.$$

We are now able to rewrite our solution as

$$(4.5) \quad g'(u) = H(u)^{-1/2} \left\{ \left(\frac{1 + 2e_3 u^3 + H(u)^{1/2}}{2} \right)^{1/3} - 2e_1 u \left(\frac{1 + 2e_3 u^3 + H(u)^{1/2}}{2} \right)^{-1/3} \right\}.$$

A moment's reflection shows that the right-hand side of (4.5) is related to the generating function of the Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$ for certain α and β .

The generating function for the Jacobi polynomials is

$$(4.6) \quad \sum_{n \geq 0} P_n^{(\alpha, \beta)}(x) t^n + R^{-1} \left(\frac{1-t+R}{2} \right)^{-\alpha} \left(\frac{1+t+R}{2} \right)^{-\beta},$$

where

$$(4.7) \quad R = (1 - 2xt + t^2)^{1/2}$$

and α and β are both greater than -1 (cf. [Sz]). Thus, if we let

$$(4.8) \quad x = (32e_1^3 - 4e_3)/4e_3 = a_3/2\sqrt{a_6}$$

and

$$(4.9) \quad t = 2e_3u^3 = \sqrt{a_6}u^3,$$

we find that

$$(4.10) \quad \begin{aligned} g'(u) &= \sum_{n \geq 0} P_n^{(0, -1/3)}(a_3/2\sqrt{a_6})(\sqrt{a_6})^n u^{3n} \\ &+ \frac{a_1}{3} \sum_{n \geq 0} P_n^{(0, 1/3)}(a_3/2\sqrt{a_6})(\sqrt{a_6})^n u^{3n+1}. \end{aligned}$$

Thus,

$$(4.11) \quad \varphi_3(CP^{3n}) = (\sqrt{a_6})^n P_n^{(0, -1/3)}(a_3/2\sqrt{a_6}),$$

$$(4.12) \quad \varphi_3(CP^{3n+1}) = \frac{a_1}{3} (\sqrt{a_6})^n P_n^{(0, 1/3)}(a_3/2\sqrt{a_6})$$

and

$$(4.13) \quad \varphi_3(CP^{3n-1}) = 0.$$

Theorem 4.1 is now proved. $\square$

For level 4 the differential equations (3.1) and (4.1) become

$$(4.14) \quad \left(\frac{f'(x)}{f(x)} \right)^4 + a_1 \left(\frac{f'(x)}{f(x)} \right)^3 + a_2 \left(\frac{f'(x)}{f(x)} \right)^2 + a_4 = \frac{1}{f(x)^4} + a_8 f(x)^4$$

and

$$(4.15) \quad g'(u)^4(1 - a_4u^4 + a_8u^8) - a_2u^2g'(u)^2 - a_1ug'(u) - 1 = 0,$$

respectively.

By making substitutions, as in the level 2 and 3 case, into (4.14) we obtain:

$$\begin{aligned} a_1 &= -8e_1, \\ a_2 &= 26e_1^2 - 6e_2, \\ e_3 &= 6e_1e_2 - 8e_1^3 \quad (\text{from } a_3 = 0), \\ a_4 &= 9e_2^2 + 30e_1^2e_2 - 83e_1^4, \\ a_8 &= 16e_1^2(3e_2 - 4e_1^2)^3. \end{aligned}$$

From (4.15) with a_1, a_2, a_4 , and a_8 expressed in terms of the e_i , we find that

$$\begin{aligned}\varphi_4(CP^1) &= -2e_1, \\ \varphi_4(CP^2) &= \frac{3}{2}(3e_1^2 - e_2), \\ \varphi_4(CP^3) &= 0, \\ \varphi_4(CP^4) &= \frac{1}{8}(-197e_1^4 + 54e_1^2e_2 + 27e_2^2).\end{aligned}$$

We notice that for level 4 we are again obtaining polynomials but this time in terms of e_1 and e_2 .

Lemma 2.1 tells us that $\varphi_4(CP^{4k+3}) = 0$, $k \geq 0$. This enables us to write the derivative of the level 4 logarithm as

$$g'(u) = \sum_{k \geq 1} \varphi_4(CP^{4k})u^{4k} + \sum_{k \geq 0} \varphi_4(CP^{4k+1})u^{4k+1} + \sum_{k \geq 0} \varphi_4(CP^{4k+2})u^{4k+2}.$$

Thus, the logarithm can be written as a sum of three expressions with coefficients which appear to be polynomials in e_1 and e_2 . These observations along with the level 2 and 3 results lead to the following conjecture.

Conjecture 4.2. *The level N logarithm can be written as a sum of $N-1$ expressions in terms of Jacobi polynomials, $P_n^{(\alpha_i, \beta_i)}(x)$, $i = 1, 2, \dots, N-1$, where α_i and β_i are suitable elements in $\{0, \pm 1/N, \pm 2/N, \dots, \pm(N-1)/N\}$.*

In [Sz, p. 68], it is shown that Jacobi polynomials can be written very nicely as

$$(4.16) \quad P_n^{(\alpha, \beta)}(x) = \frac{1}{2^n} \sum_{m=0}^n \binom{n+\alpha}{m} \binom{n+\beta}{n-m} (x-1)^{n-m} (x+1)^m.$$

If we use the substitutions $a_1 = -6e_1$, $a_3 = 32e_1^3 - 4e_3$, and $a_6 = 4e_3^2$, we can rewrite the images under the level 3 elliptic genus of the complex projective spaces via (4.16) as

$$(4.17) \quad \varphi_3(CP^{3n}) = \sum_{m=0}^n \binom{n}{m} \binom{n-\frac{1}{3}}{n-m} (8e_1^3 - 2e_3)^{n-m} (8e_1^3)^m$$

and

$$(4.18) \quad \varphi_3(CP^{3n+1}) = -2e_1 \sum_{m=0}^n \binom{n}{m} \binom{n+\frac{1}{3}}{n-m} (8e_1^3 - 2e_3)^{n-m} (8e_1^3)^m.$$

In the following proposition we let $M_*(\Gamma_1(N))$ denote the ring of modular forms for the congruence subgroup $\Gamma_1(N)$.

Proposition 4.3 (Landweber). *For level 2, e_2 and e_4 are algebraically independent and $M_*(\Gamma_1(2)) = C[e_2, e_4]$. For level 3, e_1 and e_3 are algebraically independent and, as for level 2, $M_*(\Gamma_1(3)) = C[e_1, e_3]$. Likewise, for $N = 4$, e_1 and e_2 are algebraically independent and $M_*(\Gamma_1(4)) = C[e_1, e_2]$.*

Proof. We will first prove algebraic independence and then verify that these pairs of modular forms are generators for $M_*(\Gamma_1(N))$ for $N = 2, 3$, and 4 .

For level 2 it will be enough to show that $e_4^\alpha, e_4^{\alpha-1}e_2^2, \dots, e_2^{2\alpha}$ are linearly independent for all integers $\alpha > 0$. By comparing constant terms of q -expansions

and using a standard argument, linear independence follows easily. To prove the algebraic independence assertion for levels 3 and 4, we use an identical method.

By applying several theorems in [Sh] one can conclude that

$$(4.19) \quad \dim M_k(\Gamma_1(N)) = \begin{cases} 1 + [\frac{k}{4}] & \text{for } N = 2 \text{ and for all even } k \geq 0, \\ 1 + [\frac{k}{3}] & \text{for } N = 3 \text{ and for all } k \geq 0, \\ 1 + [\frac{k}{2}] & \text{for } N = 4 \text{ and for all } k \geq 0, \end{cases}$$

where $M_k(\Gamma_1(N))$ denotes the space of modular forms of weight k and $[x]$ denotes the greatest integer less than x function.

By viewing $C[e_2, e_4] \subseteq M_*(\Gamma_1(2))$ as a graded ring with respect to weights of modular forms, we can give $C[e_2, e_4]$ a basis of monomials such that the dimensions of the space generated by the weight k basis monomials (k even) is precisely the dimension of $M_k(\Gamma_1(2))$. We therefore conclude that the above inclusion becomes an equality.

For levels 3 and 4 we again use a dimensionality argument to get the desired result. $\square$

Claim 4.4. *The identities (4.17) and (4.18) imply that*

$$\varphi_3(CP^n) \in Z\left[\frac{1}{3}\right][2e_1, 2e_3].$$

Proof. It is enough to show that all the binomial coefficients $\binom{n-\frac{1}{3}}{k}$ and $\binom{n+\frac{1}{3}}{k}$ are contained in $Z[\frac{1}{3}]$.

To show this we first note that, by solving $((1+x)^{1/3})^3 = 1+x$ recursively for the coefficients of $(1+x)^{1/3}$, we have

$$(4.20) \quad (1+x)^{1/3} = \sum_k \binom{\frac{1}{3}}{k} x^k \in Z\left[\frac{1}{3}\right][[x]].$$

This immediately implies that also

$$(4.21) \quad (1+x)^{-1/3} = \left(1 + \sum_{n>0} (-1)^n x^n\right)^{1/3} \in Z\left[\frac{1}{3}\right][[x]].$$

Now $(1+x)^n \in Z[x]$, and thus,

$$(1+x)^{n\pm 1/3} = (1+x)^n (1+x)^{\pm 1/3} = \sum_k \binom{n \pm \frac{1}{3}}{k} x^k \in Z\left[\frac{1}{3}\right][[x]].$$

Hence, $\binom{n\pm\frac{1}{3}}{k} \in Z[\frac{1}{3}]$. $\square$

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