A RELATION BETWEEN NÖRLUND AND ABEL SUMMABILITIES

MEHMET A. SARIGÖL

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ABSTRACT. This paper gives a necessary and sufficient condition in order that every series summable by a Nörlund method with $p_n = o(P_n)$ as $n \rightarrow \infty$ should be summable by Abel's method to the same sum and so extends two theorems of Zygmund.

1. INTRODUCTION

Let z designate a complex variable; then the series $\sum a_n$, with partial sums s_n , is summable to s by Abel's method if

$$(1) \quad \lim_{z \rightarrow 1} \sum_{n=0}^{\infty} a_n z^n = \lim_{z \rightarrow 1} (1-z) \sum_{n=0}^{\infty} s_n z^n = s$$

as $z \rightarrow 1$ along any path lying between two chords of the unit circle which pass through $z = 1$ [3].

Let (p_n) denote a sequence of complex numbers such that $P_n = p_0 + p_1 + \cdots + p_n \neq 0$ for all $n \geq 0$. The series $\sum a_n$ is Nörlund summable (N, p_n) to s if the sequence-to-sequence

$$(2) \quad t_n = \sum_{v=0}^n p_{n-v} P_n^{-1} s_v$$

$\rightarrow s$ as $n \rightarrow \infty$. The case $p_n = A_n^{\alpha-1}$, where $A_n^{\alpha-1}$ is defined from the identity

$$\sum_{n=0}^{\infty} A_n^{\alpha-1} x^n = (1-x)^{-\alpha},$$

is Cesàro summability (C, α) (see [1, 4]). The inverse transformation to (2) is given by

$$(3) \quad s_n = \sum_{v=0}^n c_{n-v} P_v t_v$$

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where (c_n) is defined by the formal power series relation

$$(4) \quad \sum_{n=0}^{\infty} c_n x^n \cdot \sum_{v=0}^{\infty} p_v x^v = c(x) \cdot p(x) = 1.$$

We write

$$(5) \quad P(z) = \sum_{n=0}^{\infty} P_n z^n \quad \text{and} \quad P^*(z) = \sum_{n=0}^{\infty} |P_n| |z^n|.$$

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We now prove the following theorem.

Theorem A. *Let (N, p_n) be a Nörlund method (not necessarily regular) such that*

$$(6) \quad p_n = o(P_n) \quad \text{as } n \rightarrow \infty.$$

Then so that every series Nörlund summable (N, p_n) is Abel summable to the same sum it is necessary and sufficient that

$$(7) \quad \begin{aligned} (i) \quad & |P(z)| \rightarrow \infty \text{ as } z \rightarrow 1, \text{ and} \\ (ii) \quad & P^*(z) = O(|P(z)|) \text{ as } z \rightarrow 1, \end{aligned}$$

where z approaches $z = 1$ in the angular region described.

Proof of the theorem. Suppose $t_n \rightarrow s$ as $n \rightarrow \infty$, where t_n is the (N, p_n) mean of $\sum a_n$. It follows from (6) that the series in (5) and

$$c(z) = \sum_{n=0}^{\infty} c_n z^n$$

are convergent for all $|z| < 1$, where the c_n 's are related by (4). Also because of convergence of (t_n) , $\sum_{n=0}^{\infty} P_n t_n z^n$ are absolutely convergent for $|z| < 1$. Now let (z_n) be any sequence of points within the unit circle and the angle, tending to 1. Then by (3) and (4),

$$\begin{aligned} f(z_n) &= (1 - z_n) \sum_{v=0}^{\infty} s_v z_n^v = (1 - z_n) \sum_{v=0}^{\infty} \left(\sum_{i=0}^v c_{v-i} P_i t_i \right) z_n^v \\ &= (1 - z_n) \sum_{v=0}^{\infty} P_v t_v z_n^v \cdot \sum_{i=0}^{\infty} c_i z_n^i = (1 - z_n) (p(z_n))^{-1} \sum_{v=0}^{\infty} P_v t_v z_n^v. \end{aligned}$$

On the other hand, since $(1 - z)P(z) = p(z)$ for $|z| < 1$, we have

$$f(z_n) = (P(z_n))^{-1} \sum_{v=0}^{\infty} P_v t_v z_n^v, \quad |z| < 1;$$

i.e., $(f(z_n))$ is $T = (a_{nv})$ -transform sequence of (t_n) , where

$$a_{nv} = \frac{P_v z_n^v}{P(z_n)}$$

for all n, v . Hence, it suffices to show that $f(z_n) \rightarrow s$ as $n \rightarrow \infty$ if and only if the matrix T is regular. Under the sufficiency hypothesis of the theorem, $\lim_{n \rightarrow \infty} a_{nv} = 0$ for each v , since

$$\lim_{n \rightarrow \infty} a_{nv} = P_v \lim_{n \rightarrow \infty} (P(z_n))^{-1} = 0,$$

by (7i). Moreover, we have $\sum_{v=0}^{\infty} a_{nv} = (P(z_n))^{-1} \sum_{v=0}^{\infty} P_v z_n^v = 1$ and finally

$$\sum_{v=0}^{\infty} |a_{nv}| = (|P(z_n)|)^{-1} \sum_{v=0}^{\infty} |P_v| |z_n^v| = P^*(z_n) (|P(z_n)|)^{-1} = O(1)$$

as $n \rightarrow \infty$, by (7ii). Hence $T = (a_{nv})$ is a regular matrix; it is also easy to see from the above that the conditions of the theorem are necessary.

3. APPLICATIONS

In particular, if $p_n = A_n^{\alpha-1}$, $\alpha > -1$, for all n , then $P(z) = (1-z)^{-\alpha-1}$ and $P^*(z) = (1-|z|)^{-\alpha-1}$. It is known [2] that there exists a constant M such that

$$P^*(z) \cdot (|P(z)|)^{-1} = (|1-z|(1-|z|)^{-1})^{\alpha+1} \leq M$$

for all z in the neighbourhood of the point 1. Therefore, Zygmund's results (Theorems 1.33 and 1.34 of [5]) follow from Theorem A.

On the other hand, as a corollary we have Theorem 18 of Hardy [1, p. 65] in the following form.

Theorem B. Let (N, p_n) be a Nörlund method such that $p_0 > 0$, $p_n \geq 0$, and $p_n = o(P_n)$ as $n \rightarrow \infty$. If a series is summable (N, p_n) to sum s , then (1) holds as x tends to 1 along the real axis.

Proof. Since $\sum_{n=0}^{\infty} P_n$ is divergent, it follows from Abel's theorem on power series that $P(x) \rightarrow \infty$ as $x \rightarrow 1-0$. Moreover, $P^*(x) = P(x)$ in the neighbourhood of point 1. Thus the theorem is obtained by Theorem A.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ERCIYES, 38039 KAYSERİ, TURKEY