THE H^p -BOUNDEDNESS OF THE RIESZ MEANS FOR THE EIGENFUNCTION EXPANSIONS FOR ELLIPTIC OPERATORS WITH CONSTANT COEFFICIENTS

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ABSTRACT. In this paper we study the bounded property of the Riesz means for the eigenfunction expansions for elliptic operators on the Hardy spaces. Our result generalizes the classical result due to Sjölin and Stein-Taibleson-Weiss on the Bochner-Riesz means.

1. INTRODUCTION

Let

$$a = a(D) = \sum_{|j| \leq m} a_j D^j,$$

where $j = (j_1, \dots, j_n)$, $|j| = j_1 + \dots + j_n$, and

$$D^j = (-i)^{|j|} \left(\frac{\partial}{\partial x_1} \right)^{j_1} \dots \left(\frac{\partial}{\partial x_n} \right)^{j_n},$$

be a formally selfadjoint and formally semibounded elliptic operator with constant coefficients in $\mathbb{R}^n$, that is, a_j is real and $\sum_{|j|=m} a_j \xi^j > 0$ for any $\xi = (\xi_1, \dots, \xi_n) \neq 0$. Adding a constant if necessary, we may assume that $a(\xi) \geq 0$. Let A be the unique realization of a in the Hilbert space $L^2(\mathbb{R}^n)$. Clearly, A is a positive operator. Let $A = \int_0^\infty \lambda dE_\lambda$ be the spectral resolution corresponding to A . Then E_λ is a convolution operator with the kernel

$$e_\lambda(x) = (2\pi)^{-n} \int_{a(\xi) \leq \lambda} e^{ix \cdot \xi} d\xi.$$

Therefore $E_\lambda f$ can also be defined for distributions f satisfying proper growth conditions. Certainly, $\lim_{\lambda \rightarrow \infty} E_\lambda f = f$ in $L^2(\mathbb{R}^n)$. But this is not true in $L^p(\mathbb{R}^n)$ where $1 \leq p < 2$, or $H^p(\mathbb{R}^n)$ where $0 < p \leq 1$. So we need some kind of suitable summations. In this paper we shall consider Riesz means.

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Let α be a complex; we define the Riesz means of index α (with respect to A) by

$$E_\lambda^\alpha f = \int_0^\lambda \left(1 - \frac{\mu}{\lambda}\right)^\alpha dE_\mu f.$$

We note that E_λ^α is a convolution operator with the kernel

$$e_\lambda^\alpha(x) = (2\pi)^{-n} \int_{a(\xi) \leq \lambda} \left(1 - \frac{a(\xi)}{\lambda}\right)^\alpha e^{ix \cdot \xi} d\xi.$$

If $a = -\Delta$, where Δ is the Laplace operator, then E_λ^α is none other than the Bochner-Riesz means. The properties of E_λ^α on $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, was studied by Peetre [4]. But his results are rougher than the classical results due to Stein on the Bochner-Riesz means with respect to index α (see [6]). The cause is that the pointwise estimates for the kernel e_λ^α are worse than that for the kernel of the Bochner-Riesz means. The aim of this paper is to study the bounded properties of E_λ^α on the Hardy spaces. Obviously we cannot obtain the sharp result if we use the pointwise estimates as in [7]. In consideration of this we shall use the L^1 -estimates on the set $\{x \in \mathbb{R}^n : |x| \geq r\}$ for the kernel e_λ^α and its derivatives. Our result is stated as follows.

Theorem. Let $0 < p \leq 1$, $\operatorname{Re} \alpha > \frac{n}{p} - \frac{n+1}{2}$. Then

$$\|E_\lambda^\alpha f\|_{H^p} \leq C \|f\|_{H^p}, \quad \text{for } \lambda \text{ large enough,}$$

where C is a constant independent of λ .

Corollary. Let $0 < p \leq 1$, $\operatorname{Re} \alpha > \frac{n}{p} - \frac{n+1}{2}$. Then

$$\lim_{\lambda \rightarrow \infty} E_\lambda^\alpha f = f \quad \text{in } H^p(\mathbb{R}^n).$$

The above theorem generalizes the classical result due to Sjölin [5] and Stein-Taibleson-Weiss [7] on the Bochner-Riesz means. The result is sharp even for the Bochner-Riesz means (see [7]).

2. PROOF OF THE THEOREM

Above all we recall the characterizations of $H^p(\mathbb{R}^n)$, $0 < p \leq 1$, in terms of atoms and the Riesz transforms. Let $0 < p \leq 1$, $s \geq [n(\frac{1}{p} - 1)]$. A function $f(x)$ is called a (p, ∞, s) -atom with centre x_0 if

- (i) $\operatorname{supp} f \subset B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$;
- (ii) $\|f\|_{L^\infty} \leq |B|^{-1/p}$;
- (iii) $\int f(x) x^j dx = 0$, for $0 \leq |j| \leq s$.

Lemma 1 (see [2]). Let $0 < p \leq 1$. Then $f \in H^p(\mathbb{R}^n)$ if and only if $f = \sum_{k=1}^\infty d_k f_k$ in the sense of distributions, where f_k are (p, ∞, s) -atoms and $\sum_{k=1}^\infty |d_k|^p < \infty$. Moreover,

$$\|f\|_{H^p} \sim \inf \left\{ \left(\sum_{k=1}^\infty |d_k|^p \right)^{1/p} : f = \sum_{k=1}^\infty d_k f_k \text{ is an atomic decomposition of } f \right\}.$$

We denote the Riesz transforms by R_l , $l = 1, \dots, n$, i.e.,

$$(R_l f)^\wedge(\xi) = i \frac{\xi_l}{|\xi|} \hat{f}(\xi).$$

Let $L = (l_1, \dots, l_t)$ where $0 \leq l_1, \dots, l_t \leq n$. Set $R_L = R_{l_1} \cdots R_{l_t}$ where $R_0 = I$.

Lemma 2 (see [1]). Let $\frac{n-1}{n-1+t} < p \leq 1$, $t \in \mathbb{N}$, and $f \in L^2(\mathbb{R}^n)$. Then $f \in H^p(\mathbb{R}^n)$ if and only if $R_L f \in L^p(\mathbb{R}^n)$ for all $L = (l_1, \dots, l_t)$. Moreover,

$$\|f\|_{H^p} \sim \sum_{L=(l_1, \dots, l_t)} \|R_L f\|_{L^p}.$$

By Lemmas 1 and 2, the theorem follows from the following proposition.

Proposition. Let $0 < p \leq 1$, $\operatorname{Re} \alpha > \frac{n}{p} - \frac{n+1}{2}$. Suppose f is a (p, ∞, s) -atom where $s \geq [\operatorname{Re} \alpha - \frac{n-1}{2}]$. Then we have

$$\|E_\lambda^\alpha f\|_{L^p} \leq C, \quad \text{for } \lambda \text{ large enough.}$$

Before proving the proposition, we prove the following basic lemma.

Lemma 3. For all λ large enough, we have the following estimates:

$$(i) \quad \left(\int_{|x| \geq r} |D^j e_\lambda^\alpha(x)|^q dx \right)^{1/q} \leq C_{\alpha, q} \frac{\gamma^{|j|} \lambda^{|j|/m+n/pm}}{(1 + \lambda^{1/m} r)^{\operatorname{Re} \alpha + 1/p}}, \quad 2 \leq q \leq \infty;$$

$$(ii) \quad \left(\int_{|x| \geq r} |D^j e_\lambda^\alpha(x)|^q dx \right)^{1/q} \leq C_{\alpha, q} \frac{\gamma^{|j|} \lambda^{|j|/m+n/pm}}{(1 + \lambda^{1/m} r)^{\operatorname{Re} \alpha - (n-1)/2 + n/p}}, \quad 1 \leq q < 2, \operatorname{Re} \alpha > \frac{n-1}{2},$$

where $\frac{1}{p} + \frac{1}{q} = 1$, and $C_{\alpha, q}$ and γ are constants independent of λ and j .

Proof. We prove (i) first. If $|j| = 0$, (i) is obtained by Peetre [4]. Suppose $|j| > 0$. Since a is elliptic, for λ large enough,

$$|\xi| \leq C_1 \lambda^{1/m} \quad \text{as } a(\xi) \leq \lambda,$$

where C_1 is a positive constant. Select $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that

$$\varphi(\xi) = \begin{cases} 1, & \text{as } |\xi| \leq C_1, \\ 0, & \text{as } |\xi| > C_2 > C_1. \end{cases}$$

We have

$$\begin{aligned} D^j e_\lambda^\alpha(x) &= (2\pi)^{-n} \int_{a(\xi) \leq \lambda} \left(1 - \frac{a(\xi)}{\lambda}\right)^\alpha \xi^j \varphi(\lambda^{-1/m} \xi) e^{ix \cdot \xi} d\xi \\ &= \lambda^{|j|/m} \int_{\mathbb{R}^n} e_\lambda^\alpha(y) \psi_{\lambda^{-1/m}}^j(x - y) dy, \end{aligned}$$

where $\psi_{\lambda^{-1/m}}^j(x) = \lambda^{n/m} \psi^j(\lambda^{1/m} x)$,

$$\psi^j(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \xi^j \varphi(\xi) e^{ix \cdot \xi} d\xi.$$

Fix k ; it is easy to get

$$(1) \quad |\psi^j(x)| \leq C\gamma^{|j|}(1+|x|)^{-k},$$

where C, γ may depend on k .

If $r \leq \lambda^{-1/m}$, from the estimate in (i) with $|j| = 0$ and (1), we get

$$(2) \quad \begin{aligned} & \left(\int_{|x| \geq r} |D^j e_\lambda^\alpha(x)|^q dx \right)^{1/q} \\ &= \lambda^{|j|/m} \left(\int_{|x| \geq r} \left| \int_{\mathbb{R}^n} e_\lambda^\alpha(y) \psi_{\lambda^{-1/m}}^j(x-y) dy \right|^q dx \right)^{1/q} \\ &\leq \lambda^{|j|/m} \|e_\lambda^\alpha\|_{L^q} \|\psi_{\lambda^{-1/m}}^j\|_{L^1} \leq C\gamma^{|j|} \lambda^{|j|/m+n/pm}. \end{aligned}$$

If $r > \lambda^{-1/m}$, we have

$$\begin{aligned} & \left(\int_{|x| \geq r} |D^j e_\lambda^\alpha(x)|^q dx \right)^{1/q} \\ &\leq \lambda^{|j|/m} \left(\int_{|x| \geq r} \left| \int_{|y| \leq r/2} e_\lambda^\alpha(y) \psi_{\lambda^{-1/m}}^j(x-y) dy \right|^q dx \right)^{1/q} \\ &\quad + \lambda^{|j|/m} \left(\int_{|x| \geq r} \left| \int_{|y| > r/2} e_\lambda^\alpha(y) \psi_{\lambda^{-1/m}}^j(x-y) dy \right|^q dx \right)^{1/q} \\ &= I_1 + I_2. \end{aligned}$$

Notice that $|x-y| \geq \frac{|x|}{2}$ as $|x| \geq r$ and $|y| \leq r$. By (1),

$$|\psi_{\lambda^{-1/m}}^j(x-y)| \leq C\gamma^{|j|} \lambda^{(n-k)/m} |x|^{-k}.$$

Taking $k = \operatorname{Re} \alpha + \frac{1}{p} + \frac{n}{q} + n$, we obtain

$$(3) \quad \begin{aligned} I_1 &\leq C\gamma^{|j|} \lambda^{(|j|+n-k)/m} \|e_\lambda^\alpha\|_{L^\infty} r^n \left(\int_{|x| \geq r} |x|^{-kq} dx \right)^{1/q} \\ &\leq C \frac{\gamma^{|j|} \lambda^{|j|/m+n/pm}}{(\lambda^{1/m} r)^{\operatorname{Re} \alpha + 1/p}}. \end{aligned}$$

Similarly, we have

$$(4) \quad I_2 \leq \lambda^{|j|/m} \|\psi_{\lambda^{-1/m}}^j\|_{L^1} \left(\int_{|y| > r/2} |e_\lambda^\alpha(y)|^q dy \right)^{1/q} \leq C \frac{\gamma^{|j|} \lambda^{|j|/m+n/pm}}{(\lambda^{1/m} r)^{\operatorname{Re} \alpha + 1/p}}.$$

Estimate (i) follows from (2)–(4).

Estimate (ii) is a corollary of (i). In fact,

$$\begin{aligned}
 & \left(\int_{|x| \geq r} |D^j e_\lambda^\alpha(x)|^q dx \right)^{1/q} \leq \sum_{k=0}^{\infty} \left(\int_{2^k r \leq |x| < 2^{k+1} r} |D^j e_\lambda^\alpha(x)|^q dx \right)^{1/q} \\
 (5) \quad & \leq C \sum_{k=0}^{\infty} \left(\int_{|x| \geq 2^k r} |D^j e_\lambda^\alpha(x)|^2 dx \right)^{1/2} (2^k r)^{n(1/q-1/2)} \\
 & \leq C \sum_{k=0}^{\infty} \frac{\gamma^{|j|} \lambda^{|j|/m+n/2m}}{(1 + \lambda^{1/m} 2^k r)^{\operatorname{Re} \alpha + 1/2}} (2^k r)^{n(1/q-1/2)}.
 \end{aligned}$$

If $r \leq \lambda^{-1/m}$ and we take k_0 such that $2^{k_0} r \leq \lambda^{-1/m} < 2^{k_0+1} r$, then we have

$$\begin{aligned}
 (6) \quad & \sum_{k=0}^{k_0} \frac{\gamma^{|j|} \lambda^{|j|/m+n/2m}}{(1 + \lambda^{1/m} 2^k r)^{\operatorname{Re} \alpha + 1/2}} (2^k r)^{n(1/q-1/2)} \leq \sum_{k=0}^{k_0} \gamma^{|j|} \lambda^{|j|/m+n/2m} (2^k r)^{n(1/q-1/2)} \\
 & \leq C \gamma^{|j|} \lambda^{|j|/m+n/2m} (2^{k_0} r)^{n(1/q-1/2)} \leq C \gamma^{|j|} \lambda^{|j|/m+n/pm},
 \end{aligned}$$

$$\begin{aligned}
 & \sum_{k=k_0+1}^{\infty} \frac{\gamma^{|j|} \lambda^{|j|/m+n/2m}}{(1 + \lambda^{1/m} 2^k r)^{\operatorname{Re} \alpha + 1/2}} (2^k r)^{n(1/q-1/2)} \\
 (7) \quad & \leq \sum_{k=k_0+1}^{\infty} \gamma^{|j|} \lambda^{|j|/m+(n/2-\operatorname{Re} \alpha-1/2)/m} (2^k r)^{n(1/q-1/2)-\operatorname{Re} \alpha-1/2} \\
 & \leq C \gamma^{|j|} \lambda^{|j|/m+(n/2-\operatorname{Re} \alpha-1/2)/m} (2^{k_0} r)^{n(1/q-1/2)-\operatorname{Re} \alpha-1/2} \\
 & \leq C \gamma^{|j|} \lambda^{|j|/m+n/pm}.
 \end{aligned}$$

If $r > \lambda^{-1/m}$, we have

$$\begin{aligned}
 & \sum_{k=0}^{\infty} \frac{\gamma^{|j|} \lambda^{|j|/m+n/2m}}{(1 + \lambda^{1/m} 2^k r)^{\operatorname{Re} \alpha + 1/2}} (2^k r)^{n(1/q-1/2)} \\
 (8) \quad & \leq C \sum_{k=0}^{\infty} \gamma^{|j|} \lambda^{|j|/m+(n/2-\operatorname{Re} \alpha-1/2)/m} (2^k r)^{n(1/q-1/2)-\operatorname{Re} \alpha-1/2} \\
 & \leq C \gamma^{|j|} \lambda^{|j|/m+(n/2-\operatorname{Re} \alpha-1/2)/m} r^{n(1/q-1/2)-\operatorname{Re} \alpha-1/2} \\
 & \leq C \frac{\gamma^{|j|} \lambda^{|j|/m+n/pm}}{(\lambda^{1/m} r)^{\operatorname{Re} \alpha-(n-1)/2+n/p}}.
 \end{aligned}$$

Estimate (ii) follows from (5)–(8). $\square$

Now we prove the proposition.

Proof of the proposition. Because E_λ^α commutes with translations, without loss of generality, we assume that the centre of f is the origin. Suppose $\operatorname{supp} f \subset$

$B(0, r)$. Then

$$\begin{aligned}
 & \int_{\mathbb{R}^n} |E_\lambda^\alpha f(x)|^p dx \\
 (9) \quad &= \int_{|x| < 2r} |E_\lambda^\alpha f(x)|^p dx + \sum_{k=1}^{\infty} \int_{2^k r \leq |x| < 2^{k+1} r} |E_\lambda^\alpha f(x)|^p dx \\
 &\leq \int_{|x| < 2r} |E_\lambda^\alpha f(x)|^p dx \\
 &\quad + C \sum_{k=1}^{\infty} (2^{k+1} r)^{n(1-p)} \left(\int_{2^k r \leq |x| < 2^{k+1} r} |E_\lambda^\alpha f(x)| dx \right)^p.
 \end{aligned}$$

When $|x| < 2r$,

$$|E_\lambda^\alpha f(x)| = \left| \int_{\mathbb{R}^n} f(\xi) e_\lambda^\alpha(x - \xi) d\xi \right| \leq \|f\|_{L^\infty} \|e_\lambda^\alpha\|_{L^1} \leq C r^{-n/p}.$$

Therefore

$$(10) \quad \int_{|x| < 2r} |E_\lambda^\alpha f(x)|^p dx \leq C.$$

If $r \geq \lambda^{-1/m}$, by (ii) of Lemma 3,

$$\begin{aligned}
 \int_{2^k r \leq |x| < 2^{k+1} r} |E_\lambda^\alpha f(x)| dx &= \int_{2^k r \leq |x| < 2^{k+1} r} \left| \int_{B(0, r)} f(\xi) e_\lambda^\alpha(x - \xi) d\xi \right| dx \\
 &\leq \|f\|_{L^1} \int_{2^{k-1} r \leq |x|} |e_\lambda^\alpha(x)| dx \leq C \frac{r^{n-n/p}}{(\lambda^{1/m} 2^k r)^{\operatorname{Re} \alpha - (n-1)/2}}.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 (11) \quad & \sum_{k=1}^{\infty} (2^{k+1} r)^{n(1-p)} \left(\int_{2^k r \leq |x| < 2^{k+1} r} |E_\lambda^\alpha f(x)| dx \right)^p \\
 &\leq C (\lambda^{1/m} r)^{-p(\operatorname{Re} \alpha - (n-1)/2)} \sum_{k=1}^{\infty} 2^{k(n-np-p\operatorname{Re} \alpha + p(n-1)/2)} \leq C.
 \end{aligned}$$

If $r < \lambda^{-1/m}$, notice that e_λ^α is real analytic. We develop $e_\lambda^\alpha(x - \xi)$ into the power series of ξ :

$$e_\lambda^\alpha(x - \xi) = \sum_{j \in \mathbb{Z}_+^n} \frac{i^{|j|} D^j e_\lambda^\alpha(x)}{j!} (-\xi)^j.$$

By the cancellation condition of atoms,

$$\int_{B(0, r)} f(\xi) e_\lambda^\alpha(x - \xi) d\xi = \int_{B(0, r)} f(\xi) \left(\sum_{|j| \geq s+1} \frac{i^{|j|} D^j e_\lambda^\alpha(x)}{j!} (-\xi)^j \right) d\xi.$$

According to (ii) of Lemma 3,

$$\begin{aligned}
 & \int_{2^k r \leq |x| < 2^{k+1} r} |E_\lambda^\alpha f(x)| dx \\
 & \leq \sum_{|j| \geq s+1} \frac{1}{j!} \int_{2^k r \leq |x| < 2^{k+1} r} \left(\int_{B(0, r)} |f(\xi)| |D^j e_\lambda^\alpha(x)| |\xi|^{|j|} d\xi \right) dx \\
 & \leq C \sum_{|j| \geq s+1} \frac{1}{j!} \gamma^{|j|} \lambda^{|j|/m} r^{n-n/p+|j|} (\lambda^{1/m} 2^k r)^{-(\operatorname{Re} \alpha - (n-1)/2)}, \\
 & \sum_{k=1}^{\infty} (2^{k+1} r)^{n(1-p)} \left(\int_{2^k r \leq |x| < 2^{k+1} r} |E_\lambda^\alpha f(x)| dx \right)^p \\
 (12) \quad & \leq C \sum_{k=1}^{\infty} 2^{k(n-np-p \operatorname{Re} \alpha + p(n-1)/2)} \sum_{|j| \geq s+1} \left(\frac{\gamma^{|j|}}{j!} \right)^p (\lambda^{1/m} r)^{p(|j| - \operatorname{Re} \alpha + (n-1)/2)} \\
 & \leq C.
 \end{aligned}$$

Combination of (9)–(12) gives the proof of the proposition. $\square$

Finally we prove the corollary.

Proof of the corollary. Set

$$H_2^{p,1} = \{g \in H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) : Ig \in H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\},$$

where Ig is the Riesz derivative of g :

$$(Ig)^\wedge(\xi) = |\xi| \hat{g}(\xi).$$

It is a known fact that $H_2^{p,1}$ is dense in $H^p(\mathbb{R}^n)$ (see [3]). The corollary follows from the theorem and the fact

$$(13) \quad \lim_{\lambda \rightarrow \infty} \|E_\lambda^\alpha g - g\|_{H^p} = 0, \quad \text{for } g \in H_2^{p,1}.$$

It remains to prove (13). Since a is elliptic, for λ large enough, there are positive constants A and B such that

$$A\lambda^{1/m} \leq |\xi| \leq B\lambda^{1/m}$$

when $a(\xi) = \lambda$. Take $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that

$$\varphi(\xi) = \begin{cases} 1, & \text{as } |\xi| \leq A/4, \\ 0, & \text{as } |\xi| \geq A/2. \end{cases}$$

Then we define $M_\lambda g$ by

$$(M_\lambda g)^\wedge(\xi) = \varphi(\lambda^{-1/m} \xi) \hat{g}(\xi), \quad \text{for } g \in H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n).$$

By the Hörmander multiplier theorem on $H^p(\mathbb{R}^n)$ (see [8]),

$$\|M_\lambda g\|_{H^p} \leq C \|g\|_{H^p}.$$

Let $g \in H_2^{p,1}$. Write

$$E_\lambda^\alpha g - g = h_1 + h_2 + h_3,$$

where

$$\begin{aligned} h_1 &= M_\lambda g - g, \\ h_2 &= E_\lambda^\alpha g - E_\lambda^\alpha(M_\lambda g), \\ h_3 &= E_\lambda^\alpha(M_\lambda g) - M_\lambda g. \end{aligned}$$

It is easy to prove (see [3])

$$(14) \quad \|h_1\|_{H^p} \leq \frac{C}{\lambda^{1/m}} \|I g\|_{H^p}.$$

Also, we have

$$(15) \quad \|h_2\|_{H^p} = \|E_\lambda^\alpha h_1\|_{H^p} \leq C \|h_1\|_{H^p} \leq \frac{C}{\lambda^{1/m}} \|I g\|_{H^p}.$$

Notice that

$$\frac{\lambda^{1/m}}{|\xi|} \left[\left(1 - \frac{a(\xi)}{\lambda} \right)_+^\alpha - \left(1 - \frac{a(0)}{\lambda} \right)_+^\alpha \right]$$

is a Hörmander multiplier. We have

$$\left\| E_\lambda^\alpha(M_\lambda g) - \left(1 - \frac{a(0)}{\lambda} \right)_+^\alpha M_\lambda g \right\|_{H^p} \leq \frac{C}{\lambda^{1/m}} \|M_\lambda(I g)\|_{H^p} \leq \frac{C}{\lambda^{1/m}} \|I g\|_{H^p}$$

and

$$\left\| \left[\left(1 - \frac{a(0)}{\lambda} \right)_+^\alpha - 1 \right] M_\lambda g \right\|_{H^p} \leq \frac{C}{\lambda} \|M_\lambda g\|_{H^p} \leq \frac{C}{\lambda} \|g\|_{H^p}.$$

Therefore,

$$(16) \quad \|h_3\|_{H^p} \leq \frac{C}{\lambda^{1/m}} \|I g\|_{H^p} + \frac{C}{\lambda} \|g\|_{H^p}.$$

Equation (13) follows from (14), (15), and (16). The proof of the corollary is completed. $\square$

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