TWIST-ROLL SPUN KNOTS

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ABSTRACT. We study 2-knots obtained from 1-knots by a combination of twist-spinning and roll-spinning and ask whether they are nontrivial. It is proved that, under a certain assumption, the resulting 2-knot is always nontrivial when the 1-knot is not a torus knot, making use of the Cyclic Surgery Theorem.

Let K be a knot in S^3 , and let $\tau^m \rho^n(K)$ denote the (m, n) twist-roll spun 2-knot in S^4 (see [L]). Zeeman [Z] showed that if $m \neq 0$ then $\tau^m(K)$ is a fibered 2-knot with closed fiber the m -fold cyclic branched covering space of S^3 branched over K . In particular, $\tau^{\pm 1}(K)$ is always trivial. However, for a nontrivial knot K , $\tau^m(K)$ is nontrivial if $|m| \neq 1$ by the generalized Smith conjecture [MB]. In [T] it is shown that $\rho^n(K)$ of nontrivial K is nontrivial for any n . One of the results for general m, n is that if K is the (p, q) torus knot then $\tau^m \rho^n(K)$ is equivalent to $\tau^{m-npq}(K)$ [L, Corollary 6.4]. Hence it is nontrivial if and only if $m - npq \neq \pm 1$.

We treat the case that K is not a torus knot.

Theorem 1. *Suppose that K is not a torus knot. Then $\tau^m \rho^n(K)$ is nontrivial if $|n| \geq 2$.*

Proof. It is easy to see that $\tau^m \rho^n(K)$ is equivalent to $\tau^{-m} \rho^{-n}(K)$, so we may assume that $m \geq 0$. If $m = 0$, $\rho^n(K)$ is nontrivial [T, Theorem 2]. We consider $m > 0$. By Litherland [L, Corollary 5.2], $\tau^m \rho^n(K)$ is a fibered knot with closed fiber $E_m(1/n)$. $E_m(1/n)$ denotes the closed 3-manifold obtained from the m -fold cyclic covering space E_m of the knot exterior E by $(1/n)$ -Dehn filling. (We parametrize the slopes on ∂E_m with respect to the induced framing from ∂E .) In case $m = 1$, we denote $E_1(s)$ by $K(s)$ after [CGLS]. Note that the canonical covering transformation g of E_m extends to a periodic automorphism of $E_m(s)$ for any slope s on ∂E_m , which may fix pointwise the core of the attached solid torus.

The commutator subgroup of $\pi_1(S^4 - \tau^m \rho^n(K))$ is isomorphic to $\pi_1 E_m(1/n)$. If $m = 1$, then $E_1(1/n)$ is just $K(1/n)$. Then, by [CGLS, Corollary 2], $K(1/n)$ is not simply-connected if $|n| \geq 2$. Therefore $\tau^1 \rho^n(K)$ is nontrivial. We assume that $m > 1$. Now suppose that $\tau^m \rho^n(K)$ is trivial. Then $E_m(1/n)$ is

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simply-connected. Let $\hat{g}$ denote the extension to $E_m(1/n)$ of the canonical covering transformation g of E_m . Let $\text{Fix}(\hat{g}^i)$ be the fixed-point set of $\hat{g}^i$.

We distinguish two cases.

Case 1. $\text{Fix}(\hat{g}^i) = \emptyset$ ($1 \leq i \leq m-1$).

In this case $\hat{g}$ generates a free Z_m -action on $E_m(1/n)$ with quotient $K(s)$ for some slope s on ∂E . In fact, it is not hard to see that $s = m/n$. (In the present setting, we have $(m, n) = 1$, and so m/n surely determines a slope.) Since the quotient map $E_m(1/n) \rightarrow K(m/n)$ is an m -fold cyclic covering, we have

$$\pi_1 K(m/n) \cong Z_m.$$

But this contradicts the fact that if K is not a torus knot, $K(m/n)$ does not have cyclic fundamental group unless $n = 0, \pm 1$ [CGLS, Corollary 1].

Case 2. $\text{Fix}(\hat{g}^i) \neq \emptyset$ for some i ($1 \leq i \leq m-1$).

In this case $\text{Fix}(\hat{g}^i)$ is the core C of the attached solid torus V . Then, by the generalized Smith conjecture [MB], C bounds a disk in $E_m(1/n)$. Therefore ∂E_m is compressible in E_m , and hence E_m is a solid torus (since E_m is irreducible). This implies that K is trivial, a contradiction.

Theorem 2. *Suppose that K is a nontrivial amphicheiral knot. Then $\tau^m \rho^n(K)$ is trivial if and only if $m = \pm 1$ and $n = 0$.*

Proof. Recall that nontrivial torus knots are not amphicheiral. The result follows from Theorem 1 and the fact that for a nontrivial amphicheiral knot K , $K(m)$ does not have cyclic fundamental group for any $m \neq 0$ [CGLS, Corollary 4].

Theorem 3. *Suppose that K is a composite knot. Then $\tau^m \rho^n(K)$ is trivial if and only if $m = \pm 1$ and $n = 0$.*

Proof. This follows from Theorem 1 and the fact that nontrivial Dehn surgery on a composite knot does not yield a manifold with cyclic fundamental group (see [Go, Lemma 7.1]).

In view of Theorem 1, $\tau^m \rho^{\pm 1}(K)$ may be trivial for a nontorus knot K . But it follows from [CGLS, Corollary 1] that $\tau^m \rho^1(K)$ and $\tau^m \rho^{-1}(K)$ cannot both be trivial. We give examples of a knot K such that $\tau^m \rho(K)$ is trivial. There are many examples of knots such that some Dehn surgeries on them yield lens spaces [BR, FS, Be, Ga]. Such knots yield trivial twist-roll spun knots as well.

Theorem 4. *Let K be a knot in S^3 . If m -Dehn surgery on K yields a lens space, then $\tau^m \rho(K)$ is trivial.*

Proof. We may assume that $m > 0$. Recall that $\tau^m \rho(K)$ is a fibered knot with closed fiber $E_m(1)$. As in the proof of Theorem 1, $E_m(1)$ is an m -fold covering space of $K(m)$, which is a lens space with fundamental group Z_m . Hence $E_m(1)$ is the universal covering space of $K(m)$, and so it must be S^3 . This implies that $\tau^m \rho(K)$ is trivial.

For instance, 18- and 19-Dehn surgeries on the $(-2, 3, 7)$ pretzel knot K yield lens spaces [FS]. Thus $\tau^{18} \rho(K)$ and $\tau^{19} \rho(K)$ are trivial.

Conversely, if $\tau^m \rho(K)$ is trivial for some m , then $E_m(1)$ is simply-connected, and hence $K(m)$ has cyclic fundamental group. But it is unknown that $K(m)$ is a lens space in general.

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