STRONG UNBOUNDEDNESS OF UNBOUNDED ANALYTIC FUNCTIONS

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ABSTRACT. It is proved that if f is an unbounded analytic function in the open unit disc $\mathbb{D}$, then there must exist a sequence (z_n) in $\mathbb{D}$ such that for every $j = 0, 1, 2, \dots$, $f^{(j)}(z_n) \rightarrow \infty$ as $n \rightarrow \infty$. This answers affirmatively a question asked by J. Langley and L. Rubel in 1984.

One of the problems raised in [1] was the following: *If $f(\cdot)$ is an unbounded analytic function in the open unit disc $\mathbb{D}$, must there exist a sequence (z_n) in $\mathbb{D}$ such that for every $j = 0, 1, 2, \dots$ $f^{(j)}(z_n) \rightarrow \infty$ as $n \rightarrow \infty$?*

The answer is "yes",¹ and we give a simple proof.

Proof. By the hypothesis, there is a sequence (z_m) in $\mathbb{D} = \{|z| < 1\}$ such that $f(z_m) \rightarrow \infty$ as $m \rightarrow \infty$. Hence, to prove the theorem, it is enough to show that if (z_m) is a sequence in $\mathbb{D}$ such that $f^{(j)}(z_m) \rightarrow \infty$ ($m \rightarrow \infty$) for $j = 0, 1, \dots, k$, then we can find a sequence (z'_m) in $\mathbb{D}$ such that $f^{(j)}(z'_m) \rightarrow \infty$ ($m \rightarrow \infty$) for $j = 0, 1, \dots, k+1$. This latter result can be proved as follows.

Assume that (z_m) is a sequence in $\mathbb{D}$ such that $f^{(j)}(z_m) \rightarrow \infty$ ($m \rightarrow \infty$) for $j = 0, 1, \dots, k$. If $(f^{(k+1)}(z_m))_{m=1}^\infty$ is unbounded, then there is a subsequence (z'_m) of (z_m) such that $f^{(j)}(z'_m) \rightarrow \infty$ ($m \rightarrow \infty$) for $j = 0, 1, \dots, k+1$. So let $(f^{(k+1)}(z_m))_{m=1}^\infty$ be bounded.

Split f into the sum $f(z) = T_m(z) + R_m(z)$, where

$$T_m(z) = f(z_m) + \frac{f'(z_m)}{1!}(z - z_m) + \dots + \frac{f^{(k)}(z_m)}{k!}(z - z_m)^k$$

is the Taylor polynomial of degree k and

$$R_m(z) = \frac{1}{k!} \int_{z_m}^z f^{(k+1)}(\zeta)(z - \zeta)^k d\zeta.$$

Differentiation gives relations

$$(1) \quad f^{(j)}(z) = T_m^{(j)}(z) + R_m^{(j)}(z)$$

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¹The authors of [1] proved this assertion under a certain growth condition on f , using Nevanlinna theory (see [2]).

and

$$(2) \quad R_m^{(j)}(z) = \frac{1}{(k-j)!} \int_{z_m}^z f^{(k+1)}(\zeta)(z-\zeta)^{k-j} d\zeta$$

($j = 0, 1, \dots, k$).

Each of $T_m^{(j)}(z_m) = f^{(j)}(z_m)$ ($j = 0, 1, \dots, k$) tends to ∞ as $m \rightarrow \infty$. Consider a sequence (β_m) such that $\beta_m > 0$, $\beta_m \rightarrow \infty$, and

$$(3) \quad \frac{f^{(j)}(z_m)}{\beta_m} \rightarrow \infty \quad (m \rightarrow \infty) \text{ for } j = 0, 1, \dots, k.$$

Now we need the following statement.

Lemma. Let $Q(\zeta)$ be a polynomial of degree l , $Q(\zeta_0) \neq 0$; suppose that $0 < \beta < |Q(\zeta_0)|$, and let E_β be the set of all points ζ such that $|Q(\zeta)| \leq \beta$. Then the visible (from the point ζ_0) angular size φ of E_β satisfies the inequality

$$\varphi \leq C \cdot l \cdot \sqrt[4]{\frac{\beta}{|Q(\zeta_0)|}},$$

C being an absolute constant. (φ is the linear Lebesgue measure of the central projection of E_β to the circle $|\zeta - \zeta_0| = 1$ from the point ζ_0 . If $l = 0$, then the right-hand side of the inequality is assumed to be 0.)

Proof of the Lemma. If $l = 0$, the assertion is obvious, so let $l \geq 1$. Without loss of generality we may assume that $\zeta_0 = 0$. Let $z_0, z_1, \dots, z_l$ be any $l+1$ points of E_β such that $|\arg z_i - \arg z_j| \geq \alpha$ (where $0 < \alpha \leq \frac{\pi}{2}$) when $i \neq j$. Then, obviously, $i \neq j$ implies $|z_i/z_j - 1| \geq \delta = \sin \alpha \geq \frac{2}{\pi} \alpha$. Considering the equalities $a_0 \cdot 1 + a_1 z_j + \dots + a_l z_j^l = Q(z_j)$ ($j = 0, 1, \dots, l$) as a system of linear algebraic equations with respect to the coefficients $a_0, a_1, \dots, a_l$ of the polynomial $Q(\zeta)$ ($a_0 = Q(0)$), we have

$$\begin{aligned} a_0 &= \begin{vmatrix} Q(z_0) & z_0 & z_0^2 & \dots & z_0^l \\ Q(z_1) & z_1 & z_1^2 & \dots & z_1^l \\ \dots & \dots & \dots & \dots & \dots \\ Q(z_l) & z_l & z_l^2 & \dots & z_l^l \end{vmatrix} \cdot \begin{vmatrix} 1 & z_0 & z_0^2 & \dots & z_0^{l-1} \\ 1 & z_1 & z_1^2 & \dots & z_1^{l-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & z_l & z_l^2 & \dots & z_l^{l-1} \end{vmatrix}^{-1} \\ &= (W_{l+1}(z_0, \dots, z_l))^{-1} (Q(z_0)z_1z_2 \dots z_l W_l(z_1, \dots, z_l) \\ &\quad - Q(z_1)z_0z_2 \dots z_l W_l(z_0, z_2, \dots, z_l) \\ &\quad + \dots + (-1)^l Q(z_l)z_0z_1 \dots z_{l-1} W_l(z_0, z_1, \dots, z_{l-1})), \end{aligned}$$

where $W_*(\dots)$ stands for a Vandermonde determinant of degree $*$. Therefore, a_0 equals a sum whose first term is

$$\begin{aligned} &Q(z_0) \cdot z_1z_2 \dots z_l \cdot \frac{1}{(z_1 - z_0)(z_2 - z_0) \dots (z_l - z_0)} \\ &= \frac{Q(z_0)}{(1 - z_0/z_1)(1 - z_0/z_2) \dots (1 - z_0/z_l)}; \end{aligned}$$

evidently, the modulus of this term is not greater than β/δ^l . Estimating the rest of the terms in the same way, we obtain

$$|a_0| \leq (l+1)\beta/\delta^l \quad \text{or} \quad \delta \leq ((l+1)\beta/|a_0|)^{1/l},$$

which implies

$$\alpha \leq \frac{\pi}{2} \delta \leq \pi(\beta/|a_0|)^{1/l}.$$

Therefore, for any $l+1$ points $z_j \in E_\beta$ ($j = 0, 1, \dots, l$) there are at least two of them, say z_k and z_l ($k \neq l$), such that the angle between the segments $[0, z_k]$ and $[0, z_l]$ is not greater than $\gamma = \pi(\beta/|a_0|)^{1/l}$.

Now consider all the subsets W of E_β such that for any two distinct points in W the angular distance between them is more than γ . Evidently, among these subsets W there is at least one maximal; denote it by W^* . The foregoing shows that W^* contains $\leq l$ points, while its maximality means that any point $z \in E_\beta$ is at an angular distance $\leq \gamma$ from some point in W^* . Consequently, E_β is contained in the union of $\leq l$ angles, the spread of each of them being equal to 2γ . The lemma is proved (with $C = 2\pi$). $\square$

Apply the lemma to the polynomials $T_m^{(j)}(\zeta)$ for $j = 0, 1, \dots, k$. We have $T_m^{(j)}(z_m) = f^{(j)}(z_m)$; hence, by (3) and the lemma, the visible (from z_m) angular size of the set X_m of all points z for which at least one of the inequalities

$$|T_m^{(j)}(z)| \leq \beta_m$$

($j = 0, 1, \dots, k$) holds tends to 0 as $m \rightarrow \infty$.

Thus for any sufficiently large m (which guarantees also that $|z_m| > \frac{1}{2}$) there exists a ray from z_m that intersects the circle $\{|z| = \frac{1}{2}\}$ and on which

$$(4) \quad |T_m^{(j)}(z)| > \beta_m \quad \text{for } j = 0, 1, \dots, k.$$

Let $\tilde{z}_m$ be the first intersection of the above ray with $\{|z| = \frac{1}{2}\}$; for m large enough we have

$$\max_{z \in [z_m, \tilde{z}_m]} |f^{(k+1)}(z)| > \beta_m/2$$

(because $|f^{(k)}(\tilde{z}_m)|$ and $|z_m - \tilde{z}_m|$ are bounded, while, by (3), $|f^{(k)}(z_m)|/\beta_m \rightarrow \infty$ as $m \rightarrow \infty$) and, furthermore,

$$|f^{(k+1)}(z_m)| < \beta_m/2.$$

Denote by z'_m the first point on $[z_m, \tilde{z}_m]$, where

$$|f^{(k+1)}(z)| = \beta_m/2.$$

Since

$$|z_m - \tilde{z}_m| \leq \sqrt{1 - (1/2)^2} < 1,$$

it follows from (2) that

$$|R_m^{(j)}(z'_m)| \leq \beta_m/2 \quad (j = 0, 1, \dots, k)$$

and from (1) and (4) that

$$|f^{(j)}(z'_m)| \geq |T_m^{(j)}(z'_m)| - \beta_m/2 > \beta_m/2 \quad (j = 0, 1, \dots, k).$$

Because

$$|f^{(k+1)}(z'_m)| = \beta_m/2,$$

we have

$$f^{(j)}(z'_m) \rightarrow \infty \quad (m \rightarrow \infty) \text{ for } j = 0, 1, \dots, k+1. \quad \square$$

The above result may be slightly improved:

Let $\varphi(r)$ be a nondecreasing positive continuous function on $[0, 1)$; suppose that for some sequence (z_n) in $\mathbb{D}$

$$f(z_n)/\varphi(|z_n|) \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

Then there exists a sequence (z_n^*) in $\mathbb{D}$ such that

$$f^{(j)}(z_n^*)/\varphi(|z_n^*|) \rightarrow \infty \quad (n \rightarrow \infty)$$

for every $j = 0, 1, 2, \dots$.

The corresponding result for entire functions is the following:

Let $f(z)$ be an entire function and $(\varphi_j(r))_{j=0}^\infty$ a sequence of nondecreasing positive continuous functions on $[0, \infty)$ such that

- (i) $\lim_{r \rightarrow \infty} (\varphi_j(Ar)/\varphi_j(r)) = \infty$, $j = 0, 1, 2, \dots$, for some constant $A > 1$;
- (ii) $r \cdot \varphi_{j+1}(r) \leq C_j \varphi_j(r)$ ($r \geq r_j$, $0 < r_j < \infty$, $C_j > 0$), $j = 0, 1, 2, \dots$;
- (iii) for some sequence (z_n) in $\mathbb{C}$

$$f(z_n)/\varphi_0(|z_n|) \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

Then there exists a sequence (z_n^*) in $\mathbb{C}$ such that

$$f^{(j)}(z_n^*)/\varphi_j(|z_n^*|) \rightarrow \infty \quad (n \rightarrow \infty)$$

for every $j = 0, 1, 2, \dots$.

We note that for any transcendental entire function, $f(z)$, there exists a sequence of functions $(\varphi_j(r))_{j=0}^\infty$ satisfying the assumptions of this statement; hence, the latter statement (whose proof in essence coincides with the above one) provides a solution to yet another problem from [1]: to give for such $f(z)$ an elementary proof of the existence of a sequence (z_n) in $\mathbb{C}$ for which

$$f^{(j)}(z_n) \rightarrow \infty \quad \text{as } n \rightarrow \infty$$

($j = 0, 1, 2, \dots$).

The above estimates are far from being exact. It would be interesting to know whether the following statement is true.

Suppose that $\varphi(\cdot) \in C^\infty(0, R)$ (where $0 < R \leq \infty$) and $\varphi^{(j)}(r) > 0$ for all $j = 0, 1, 2, \dots$, $0 < r < R$. Let $f(z)$ be an analytic function in $D_R = \{z: |z| < R\}$ such that

$$\limsup_{r \nearrow R} \left(\max_{|z|=r} |f(z)|/\varphi(r) \right) = \infty.$$

Then there exists a sequence (z_n) , $|z_n| \nearrow R$, such that

$$f^{(j)}(z_n)/\varphi^{(j)}(|z_n|) \rightarrow \infty \quad \text{as } n \rightarrow \infty$$

($j = 0, 1, 2, \dots$).

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REFERENCES

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