ON SMOOTH AND ANALYTIC DISKS IN $\mathbb{C}^2$ WITH COMMON BOUNDARY

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ABSTRACT. We construct explicitly a real analytic embedded real two-dimensional disk in $\mathbb{C}^2$ totally real except at exactly one elliptic complex tangent point, which shares the common boundary with an analytic disk in the same $\mathbb{C}^2$, but does not contain this analytic disk in its envelope of holomorphy. The same proof further yields an explicit example of a holomorphic re-embedding of the standard two-sphere into $\mathbb{C}^2$ in such a way that the new embedding shows some exceptional properties: It bounds a real three-dimensional Levi flat cell in $\mathbb{C}^2$ foliated by analytic disks, which is *not* polynomially convex. In particular, this new embedding of the standard two-sphere cannot be a subset of any compact strongly pseudoconvex surface in $\mathbb{C}^2$ or a subset of any strongly pseudoconvex graph in $\mathbb{C}^2$ in the sense of Bedford and Gaveau.

The main results of this article are the following two theorems.

Theorem 1. *There exists a C^∞ smooth, real two-dimensional smooth disk M in $\mathbb{C}^2$ which is totally real except at exactly one elliptic complex tangent point such that:*

- (a) $\partial M = \{(e^{i\theta}, 0) \in \mathbb{C}^2 \mid \theta \in \mathbb{R}\}$ and
- (b) *there exists a contractible bounded strongly pseudoconvex domain Ω with a C^∞ smooth boundary that contains M but does not contain the complex analytic disk $\{(z, 0) \in \mathbb{C}^2 \mid |z| < 1\}$. In fact, Ω can be chosen such that it is biholomorphic to a strongly convex domain.*

Theorem 2. *There exists an embedded real analytic sphere $\Sigma \subset \mathbb{C}^2$ of real dimension two which bounds a real three-dimensional Levi flat surface $\Gamma \subset \mathbb{C}^2$ and admits a complex analytic disk A such that $\partial A \subset \Sigma = \partial \Gamma$ but $A \not\subset \Gamma$. Furthermore, Σ admits exactly two complex tangent and, hence, elliptic, points.*

In the above and throughout this paper, by an *analytic disk* we mean (the image of) an injective holomorphic embedding of the open unit disk in $\mathbb{C}$ into $\mathbb{C}^2$ which extends continuously across the boundary. The smooth real disk is

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(the image of) a smooth embedding of the open unit disk in $\mathbb{C}$ into $\mathbb{C}^2$ which admits a continuous extension across the boundary.

The definition of the term *elliptic (complex tangent) point* is due to Bishop [4]. The elliptic points on a real surface are special because they initiate a family of analytic disks with boundaries in the real surface forming a Levi flat surface in its neighborhood ([1, 2, 4, 6, 7, 10, 11], et al.).

Remarks. (1) Notice that our results are in contrast to the theorems by Bedford and Gaveau [2], Bedford and Klingenberg [3], Gromov [7], and others concerning global envelopes of holomorphy.

(2) Our results, however, do *not* answer the question raised by Forstnerič in [5] since the smooth disk M and the analytic disk A in Theorem 1 meet at a common interior point. After the first version of this article was written, Forstnerič informed us that the methods presented in [5] can also prove our main result in a somewhat implicit way. However, our construction in this article is still of separate interest because it is more elementary and explicit and does not depend on the methods of [5].

Proof of Theorems 1 and 2. Denote $E^3 := \{(z_1, z_2) \in \mathbb{C}^2 \mid \text{Im } z_2 = 0\}$ and write the coordinate system of $\mathbb{C}^2 = \mathbb{R}^4$ as

$$z_1 = x_1 + \sqrt{-1}y_1 := (x_1, y_1), \quad z_2 = x_2 + \sqrt{-1}y_2 := (x_2, y_2).$$

Consequently, $(x_1, y_1; x_2)$ denotes a real coordinate system for E^3 . Consider the plane P generated by x_2 and y_1 axes and the ellipse

$$\left(\frac{x_2}{1+\eta}\right)^2 + \left(\frac{y_1}{\eta}\right)^2 = \frac{1}{\lambda} \quad (\lambda \geq 1)$$

rotated by a small angle $\alpha > 0$. Call this rotated ellipse Q . Then Q is represented by

$$\left(\frac{cx_2 - sy_1}{1+\eta}\right)^2 + \left(\frac{sx_2 + cy_1}{\eta}\right)^2 = \frac{1}{\lambda}$$

where

- (a) $c = \cos \alpha$, $s = \sin \alpha$ and
- (b) $\lambda > 1$ is chosen so that the x_2 -intercepts of Q are ± 1 .

Denote by L_c the line segment in P which is part of the line $y_1 = c$ cut by and contained in the interior of the ellipse Q . For each t choose the real numbers $c(t)$ and $r(t)$ such that

- (c) the point $(y_1, x_2) = (t, c(t))$ is the midpoint of L_t and
- (d) $2r(t) = \text{length}(L_t)$.

Then we define the ellipsoidal surface $\mathcal{E}$ contained in E^3 defined by the equation $x_1^2 + (x_2 - c(y_1))^2 = r(y_1)^2$. It is easy to check that $\mathcal{E}$ is in fact a smooth surface, and that it has exactly two points where the tangent planes are parallel to the x_1y_1 -plane. Hence, when we consider $\mathcal{E}$ embedded in $\mathbb{C}^2$ by inclusion map, these two are still the only complex tangent points on $\mathcal{E}$. It follows directly by construction that these points are in fact elliptic.

Moreover, our construction also shows that one elliptic complex tangent point belongs to the half space $\{y_1 > 0\}$, whereas the other one is in the other open half space $\{y_1 < 0\}$. Thus we let

$$S := \mathcal{E} \cap \{(x_1, y_1, x_2) \in E^3 \mid y_1 > 0\}.$$

Clearly, $\partial S = \{(x_1, 0, x_2) \in E^3 \mid x_1^2 + x_2^2 = 1\}$.

Now consider the mapping $\Phi: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ defined by

$$\Phi(z, w) = (z, (1+i)w - zw^2 - iz^2w^3)$$

which was first introduced by Wermer in [12]. Let $\tau(z, w) = (z + iw, z - iw)$. Notice that there exists an open subset U of $\mathbb{C}^2$ containing $X := \{(x_1, 0, x_2, 0) \in \mathbb{C}^2 \mid x_1^2 + x_2^2 \leq 1\}$ such that $\tilde{\Phi} := \Phi \circ \tau$ restricted on U is a biholomorphism into. Without loss of generality, we may assume that U is biholomorphic to the bidisk.

Notice that by choosing a sufficiently small $\eta > 0$ in the above we can have $\mathcal{E}$ contained in U_δ . Also notice that $\tilde{\Phi}$ maps ∂S onto the circle $\Gamma = \{(z, 0) \in \mathbb{C}^2 \mid |z| = 1\}$. Thus, we let $M = \tilde{\Phi}(S)$. $\overline{M}$ is now contained in the domain $D = \tilde{\Phi}(U_\delta)$. Notice that we can also have that $\overline{M}$ is contained in the strongly pseudoconvex domain Ω biholomorphic to a strongly convex domain by using a smooth exhaustion of the bidisk by strongly convex domains. This proves Theorem 1. Theorem 2 follows from letting $\Sigma = \tilde{\Phi}(\mathcal{E})$ and letting Γ be the image under $\tilde{\Phi}$ of the obvious real three-dimensional ball bounded by $\mathcal{E}$ in $E^3 \subset \mathbb{C}^2$.

Remarks on the interior intersection of A and M . Recall that the preimage under the map Φ of the flat analytic disk A is contained in the set represented by

$$(1+i)w - zw^2 - iz^2w^3 = w(1-zw)(1+i+izw) = 0.$$

With a sufficiently small δ for the domain D in the previous section, the variety defined by $1+i+izw = 0$ does not meet D . Therefore, we must only look at $w(1-zw) = 0$. Consequently, the preimage of A under the map $\tilde{\Phi} = \Phi \circ \tau$ lies in the algebraic variety, say V , in $\mathbb{C}^2$ defined by $(z_1 - iz_2)(1 - z_1^2 - z_2^2) = 0$. We now restrict our attention to the real three-dimensional subspace F^3 given by $\text{Im } z_2 = 0$. It is easy to see that any surface in E^3 homeomorphic to the closed unit disk in $\mathbb{C}$ with the boundary $\{x_1^2 + x_2^2 = 1, y_1 = 0\}$ must intersect the line defined by $x_1 = 0, x_2 + y_1 = 0$. Moreover, the disks A and M we constructed in the preceding section share exactly two interior intersection points.

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