ALL CLUSTER POINTS OF COUNTABLE SETS IN SUPERCOMPACT SPACES ARE THE LIMITS OF NONTRIVIAL SEQUENCES

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ABSTRACT. A space is called supercompact if it has an open subbase such that every cover consisting of elements of the subbase has a subcover consisting of two elements. In this paper we prove that, in a continuous image of a closed G_δ -set of a supercompact space, a point is a cluster point of a countable set if and only if it is the limit of a nontrivial sequence. As corollaries, we answer questions asked by J. van Mill et al.

In this paper we prove that, in a continuous image of a closed G_δ -set of a supercompact space, any cluster point of a countable set is the limit of a nontrivial convergent sequence. As corollaries, we answer two questions asked by van Douwen and van Mill and by van Mill and Mills in [6] and [9], respectively.

Every space in the present paper is assumed to be Hausdorff, and the word "subbase" is assumed to mean "subbase for closed sets". For a space X a family $\mathcal{F}$ of closed sets is called *linked* if $F \cap F' \neq \emptyset$ for all $F, F' \in \mathcal{F}$, and *binary* if every linked subfamily has nonempty intersection. A space is called *supercompact* if it has a binary subbase [7]. Many compact spaces, but not all, are supercompact. For example, all compact metric spaces are supercompact [5, 11]; all continuous images of compact ordered spaces are supercompact [4]. On the other hand, closed G_δ -sets of supercompact spaces are neither supercompact in general [3] nor continuous images of supercompact spaces [10]. Moreover, Bell gave an example of a nonsupercompact dyadic space (= a continuous image of 2^κ) [2].

In 1982 van Douwen and van Mill proved in [6] that, in a continuous image of a supercompact space, at least one cluster point of a countable set is the limit of a nontrivial convergent sequence in the whole space; at most countably many cluster points are not so. The result suggested to them the following question.

Question 1 [6]. Let Y be a continuous image of a supercompact space (or just a supercompact space). If K is a countable subset of Y , then is every cluster point of K the limit of a nontrivial convergent sequence?

Applying the result mentioned above, van Mill and Mills proved in [9] that,

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under a set-theoretical hypothesis, every infinite continuous image of a closed G_δ -set of a supercompact space contains a nontrivial convergent sequence. Then, they asked if the set-theoretical hypothesis may be dropped.

Question 2 [9]. If Y is an infinite continuous image of a closed G_δ -set of a supercompact space, then does Y contain a nontrivial convergent sequence?

In this paper we prove the next theorem, which answers the above two questions affirmatively.

Theorem. *Let Y be a continuous image of a closed G_δ -subset of a supercompact space and K a countable subset of Y . Then every cluster point of K is the limit of a nontrivial convergent sequence in Y .*

To show the theorem, we first give some lemmas. The first two lemmas can be directly proved. Let N be the set of all natural numbers.

Lemma 1. *Let $f: X \rightarrow Y$ be a continuous mapping from a compact space X onto a space Y and $\{A_n \subset X: n \in N\}$ a decreasing sequence of closed sets of X . If $\bigcap_{n \in N} A_n \subset f^{-1}(y)$ for some $y \in Y$, then $f(a_n) \rightarrow y$ for any $a_n \in A_n$.*

Now let $\mathcal{S}$ be a subbase (note that subbase means subbase for closed subsets) for a compact space X . Let $\mathcal{S}^\cap = \{\bigcap \mathcal{S}_0: \mathcal{S}_0 \subset \mathcal{S}\}$. Then $\mathcal{S}^\cap$ is also a subbase. Furthermore, if $\mathcal{S}$ is binary, then $\mathcal{S}^\cap$ is also binary. Thus we assume throughout this paper that every subbase is closed with respect to arbitrary intersections. Now we fix a point $p \in X$. For $A \subset X$ let

$$J(A) = \bigcap \{S \in \mathcal{S}: p \in S \text{ and } S \cap A \neq \emptyset\}.$$

If $A = \{a\}$, we write $J(a)$ instead of $J(\{a\})$.

Lemma 2. *Let $\mathcal{S}$ be a subbase for a compact space X , F a closed subset, and U an open subset of X . If $F \subset U$, then there exist $S_1, S_2, \dots, S_n \in \mathcal{S}$ such that $F \subset S_1 \cup S_2 \cup \dots \cup S_n \subset U$. In particular, if a point $x \in U$, then there exist $S_1, S_2, \dots, S_n \in \mathcal{S}$ such that $x \in S_1 \cap S_2 \cap \dots \cap S_n$ and $x \in \text{int}(S_1 \cup S_2 \cup \dots \cup S_n) \subset S_1 \cup S_2 \cup \dots \cup S_n \subset U$.*

Lemma 3. *Let $A, B \subset X$. If $S \cap A \neq \emptyset$ implies $S \cap B \neq \emptyset$ for every $S \in \mathcal{S}$ with $p \in S$, then $p \in \overline{A}$ implies $p \in \overline{B}$. In particular, if $p \in \overline{A}$, then $J(A) = \{p\}$.*

Proof. Let $p \in \overline{A}$. If $p \notin \overline{B}$, then by Lemma 2 there exist $S_1, S_2, \dots, S_n \in \mathcal{S}$ such that $p \in S_1 \cap S_2 \cap \dots \cap S_n$ and

$$(1) \quad p \in \text{int}(S_1 \cup S_2 \cup \dots \cup S_n) \subset S_1 \cup S_2 \cup \dots \cup S_n \subset X \setminus \overline{B}.$$

Since $p \in \overline{A}$, there exists S_i such that $S_i \cap A \neq \emptyset$. Hence $S_i \cap B \neq \emptyset$, which contradicts (1). Now if $q \in J(A)$, then $S \cap A \neq \emptyset$ implies $q \in S$ for every $S \in \mathcal{S}$ with $S \ni p$. Thus we have $p \in \overline{\{q\}} = \{q\}$; that is, $J(A) = \{p\}$.

Lemma 4. *Let $E, Z \subset X$ be closed sets and $C = \{c_n: n \in N\} \subset Z$ a countable set. If $p \in E \cap \overline{C}$ and $E \cap C = \emptyset$, then one of the following statements holds.*

(A) *There exists an increasing sequence $\{A_n: n \in N\}$ of subsets of C such that $Z \cap J(A_n) \not\subset E$ for all $n \in N$ but $Z \cap \bigcap_{n \in N} J(A_n) \subset E$.*

(B) *There exists a sequence $\{A_n: n \in N\}$ of subsets of C such that $C = \bigcup_{n \in N} A_n$ and $Z \cap J(A_n) \not\subset E$ but $Z \cap J(A_n) \cap J(A_m) \subset E$ for all $n, m \in N$, $n \neq m$.*

Proof. Suppose that there exists no sequence of subsets of C satisfying the conditions in (A). Then we construct $\{A_n: n \in N\}$ such that, for all $n \in N$ and $c \in C \setminus \bigcup_{i \leq n} A_i$,

$$Z \cap J(A_n) \not\subset E, \quad Z \cap J(A_n) \cap J(c) \subset E, \\ c_{k(n)} \in A_n \subset C \setminus \bigcup_{i < n} A_i,$$

where $k(n)$ is the least k satisfying $c_k \in C \setminus \bigcup_{i < n} A_i$.

In fact, if $\{A_i: i < n\}$ have been defined satisfying the required conditions, then $\bigcup_{i < n} A_i \neq C$, since otherwise, $p \in \overline{A_i}$ for some $i < n$. Hence, Lemma 3 implies that $Z \cap J(A_i) \subset J(A_i) = \{p\} \subset E$, which contradicts the assumption. Since C is countable, $c_{k(n)} \in C \setminus (E \cup \bigcup_{i < n} A_i)$, and (A) does not hold, there exists a maximal subset A_n of $C \setminus \bigcup_{i < n} A_i$ such that $c_{k(n)} \in A_n$ and $Z \cap J(A_n) \not\subset E$. Then for all $c \in C \setminus \bigcup_{i \leq n} A_i$ we have $Z \cap J(A_n) \cap J(c) \subset E$. The inductive definition is completed. It is clear that the sequence $\{A_n: n \in N\}$ satisfies the required conditions in (B).

Proof of the theorem. Suppose that Y and $K \subset Y$ satisfy the conditions in Theorem and $y \in \overline{K} \setminus K$. Let X be a supercompact space with a binary subbase $\mathcal{S}$ and $Z \subset X$ a closed G_δ -set, and let $f: Z \rightarrow Y$ be a continuous mapping from Z onto Y . Then there exist a countable set $C \subset Z$ and $p \in Z$ such that $f(C) = K$ and $p \in \overline{C} \cap f^{-1}(y)$. Clearly, $E = f^{-1}(y)$ and Z and C satisfy the requests in the last lemma. Hence there exists a sequence $\{A_n: n \in N\}$ of subsets of C satisfying the conditions in (A) or (B). Choose $z_n \in Z \cap J(A_n) \setminus f^{-1}(y)$. Then $\{f(z_n): n \in N\}$ is a sequence in Y , and $f(z_n) \neq y$ for all $n \in N$. If (A) holds, then Lemma 1 implies $f(z_n) \rightarrow y$. If (B) holds, then by Lemma 3 we have that

$$(2) \quad p \in \overline{\{z_n: n \in N\}},$$

$$(3) \quad Z \cap J(z_n) \cap J(z_m) \subset f^{-1}(y)$$

for all $n \neq m$. To complete the proof of the theorem, it suffices to show the following lemma.

Lemma 5. *If $D = \{z_n: n \in N\} \subset Z \setminus f^{-1}(y)$ satisfies (2) and (3), then there exists a subsequence $\{z_{n_k}, k \in N\}$ of $\{z_n, n \in N\}$ such that $f(z_{n_k}) \rightarrow y$.*

Proof. Since Z is a G_δ -set, we can put $Z = \bigcap \{U_k: k \in N\}$ for open subsets U_k ($k \in N$) of X with $U_{k+1} \subset U_k$. Then by Lemma 2 for every $k \in N$ there exist $S_1, S_2, \dots, S_m \in \mathcal{S}$ such that

$$p \in S_1 \cap S_2 \cap \dots \cap S_m$$

and

$$p \in \text{int}(S_1 \cup S_2 \cup \dots \cup S_m) \subset S_1 \cup S_2 \cup \dots \cup S_m \subset U_k.$$

Since $p \in \overline{D}$, there exists S_i such that $S_i \cap D$ is infinite. Thus $\{n: J(z_n) \subset U_k\}$ is infinite for $k \in N$. Therefore, we can inductively define $\{n_k: k \in N\}$ such that $n_1 < n_2 < \dots$ and for $k \in N$

$$(4) \quad J(z_{n_k}) \subset U_k.$$

Then $f(z_{n_k}) \rightarrow y$. In fact, otherwise there exists an open set $V \ni y$ in Y such that $\{k: f(z_{n_k}) \notin V\}$ is infinite. It follows from $f^{-1}(y) \subset f^{-1}(V)$ and Lemma 2 that there exist $T_1, T_2, \dots, T_m \in \mathcal{S}$ such that

$$(5) \quad X \setminus f^{-1}(V) \subset T_1 \cup T_2 \cup \dots \cup T_m \subset X \setminus f^{-1}(y).$$

Since $\{k: f(z_{n_k}) \notin V\}$ is infinite, there exists T_i such that $\{k: z_{n_k} \in T_i\}$ is infinite. Thus (4) implies that

$$\bigcap \{J(z_{n_k}): z_{n_k} \in T_i\} \subset \bigcap \{U_k: z_{n_k} \in T_i\} = Z.$$

Hence it follows from (3) and (5) that

$$\begin{aligned} T_i \cap \bigcap \{J(z_{n_k}): z_{n_k} \in T_i\} \\ = T_i \cap Z \cap \bigcap \{J(z_{n_k}): z_{n_k} \in T_i\} \subset T_i \cap f^{-1}(y) = \emptyset. \end{aligned}$$

On the other hand, the family

$$\{T_i\} \cup \{J(z_{n_k}): z_{n_k} \in T_i\}$$

is a linked subfamily of $\mathcal{S}$. Hence,

$$T_i \cap \bigcap \{J(z_{n_k}): z_{n_k} \in T_i\} \neq \emptyset$$

since $\mathcal{S}$ is binary (this is the only point in the proof where we use the fact that $\mathcal{S}$ is binary). Now a contradiction occurs.

Remark. For a nonisolated point $y \in Y$ let

$$t(y) = \min\{|A|: A \subset Y \setminus \{y\} \text{ and } \overline{A} \ni y\}.$$

In Theorem we proved that in certain spaces Y if $t(y)$ is countable, then y is the limit of a nontrivial sequence in Y . In fact, it is not difficult to extend the result to a general case. We call $Z \subset X$ to be a G_μ -set if $Z = \bigcap \{U_\xi: \xi < \mu\}$ for a decreasing open family $\{U_\xi: \xi < \mu\}$. Then we have

Let Y be a continuous image of a closed G_μ -set of a supercompact space and $y \in Y$ a nonisolated point. If $\mu \leq \text{cf}(t(y))$, then y is the limit of a nontrivial α -sequence in Y for some limit ordinal $\alpha \leq t(y)$.

From this statement the following corollary is obtained.

Corollary. *If Y is a continuous image of a supercompact space, then every nonisolated point in Y is the limit of a nontrivial linear net.*

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