APPROXIMATION METHOD AND EQUILIBRIA OF ABSTRACT ECONOMIES

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(Communicated by Dale Alspach)

ABSTRACT. By applying an equilibrium existence theorem for a qualitative game due to Ding and Tan and by employing an "approximation" method used by Tulcea, we obtain an equilibrium existence theorem for an abstract economy (generalized game) in which the constraint correspondences are not assumed to have open graphs nor open lower sections (which are generally assumed in the literature). Our result generalizes the corresponding results of Shafer and Sonnenschein (1975), Borglin and Keiding (1976), Yannelis and Prabhakar (1983), Tulcea (1986), and Chang (1990) in several ways.

1. INTRODUCTION

Let E be a vector space and $A \subset E$. We shall denote by $\text{co } A$ the convex hull of A . If A is a subset of a topological space X , the closure of A in X is denoted by $\text{cl}_X A$ and the interior of A in X is denoted by $\text{int}_X(A)$. Also, A is said to be compactly open in X if, for each nonempty compact subset C of X , $A \cap C$ is open in C . Let X be a set; we shall denote by 2^X the family of all subsets of X . Let X and Y be sets and $F : X \rightarrow 2^Y$. Then the graph of F , denoted by $\text{Graph}(F)$, is the set $\{(x, y) \in X \times Y : y \in F(x)\}$. Suppose X and Y are topological spaces and $F : X \rightarrow 2^Y$. Then

- (1) F is said to have an open graph if $\text{Graph}(F)$ is open in $X \times Y$;
- (2) the map $\bar{F} : X \rightarrow 2^Y$ is defined by

$$\bar{F}(x) = \{y \in Y : (x, y) \in \text{cl}_{X \times Y} \text{Graph}(F)\}$$

for each $x \in X$ (the set $\text{cl}_{X \times Y} \text{Graph}(F)$ is called the adherence of the graph of T);

- (3) F is said to be lower semicontinuous (respectively, upper semicontinuous) if for each closed (respectively, open) subset C of Y the set $\{x \in X : F(x) \subset C\}$ is closed (respectively, open) in X ; and

Received by the editors January 12, 1993.

1991 *Mathematics Subject Classification*. Primary 47H04, 47N10, 49J45, 90A14, 90D06, 90D13.

Key words and phrases. Compactly open, open graph, closed graph, lower semicontinuous, upper semicontinuous, class L_C , L_C -majorant, L_C -majorized, one-person game, qualitative game, generalized game, abstract economy, equilibrium point, paracompact, topological vector space, locally convex space.

(4) F is said to have open lower sections if $F^{-1}(y) := \{x \in X : y \in F(x)\}$ is open in X for each $y \in Y$.

Let X be a topological space, Y a nonempty subset of a vector space E , $\theta : X \rightarrow E$ a map, and $\phi : X \rightarrow 2^Y$ a correspondence. Then

(1) ϕ is said to be of class $L_{\theta, C}$ if, for every $x \in X$, $\text{co } \phi(x) \subset Y$ and $\theta(x) \notin \text{co } \phi(x)$ and, for each $y \in Y$, $\phi^{-1}(y) = \{x \in X : y \in \phi(x)\}$ is compactly open in X ; and

(2) a correspondence $\phi_x : X \rightarrow 2^Y$ is said to be an $L_{\theta, C}$ -majorant of ϕ at $x \in X$ if there exists an open neighborhood N_x of x in X such that

- (a) for each $z \in N_x$, $\phi(z) \subset \phi_x(z)$ and $\theta(z) \notin \text{co } \phi_x(z)$,
- (b) for each $z \in X$, $\text{co } \phi_x(z) \subset Y$, and
- (c) for each $y \in Y$, $\phi_x^{-1}(y)$ is compactly open in X ;

(3) ϕ is $L_{\theta, C}$ -majorized if for each $x \in X$ with $\phi(x) \neq \emptyset$ there exists an $L_{\theta, C}$ -majorant of ϕ at x in X .

Our notions of correspondences being of class $L_{\theta, C}$, $L_{\theta, C}$ -majorant of ϕ at x , and $L_{\theta, C}$ -majorized generalize the corresponding notions of correspondences being of class L , $\mathcal{L}$ -majorant of ϕ at x , and $\mathcal{L}$ -majorized introduced by Yannelis and Prabhakar [10, p. 240]. As in [4], we shall only deal with either the case (I) $X = Y$ and is a nonempty convex subset of a topological vector space and $\theta = I_X$, the identity map on X , or the case (II) $X = \prod_{i \in I} X_i$ and $\theta = \pi_j : X \rightarrow X_j$ is the projection of X onto X_j and $X_j = Y$ is a nonempty convex subset of a topological vector space. In both cases (I) and (II), we shall write L_C in place of $L_{\theta, C}$.

A one-person game is a quadruple $(X; A, B; P)$ where X is a topological space, $A, B : X \rightarrow 2^X$ are constraint correspondences, and $P : X \rightarrow 2^X$ is a preference correspondence. An equilibrium point for $(X; A, B; P)$ is a point $x^* \in X$ such that $x^* \in \overline{B}(x^*)$ and $A(x^*) \cap P(x^*) = \emptyset$.

Let I be a (finite or infinite) set of agents (players). An abstract economy (generalized game) is a family $\Gamma = (X_i; A_i, B_i; P_i)_{i \in I}$ of quadruples $(X_i; A_i, A_i; P_i)$ where, for each $i \in I$, X_i is a topological space, $A_i, B_i : X := \prod_{j \in I} X_j \rightarrow 2^{X_i}$ are constraint correspondences, and $P_i : X \rightarrow 2^{X_i}$ is a preference correspondence. An equilibrium point for Γ is a point $x^* \in X$ such that, for each $i \in I$, $x_i^* = \pi_i(x^*) \in \overline{B_i}(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$ where $\pi_i : X \rightarrow X_i$ is the projection. We remark that when $\overline{B_i}(x^*) = \text{cl}_{X_i} B_i(x^*)$ (which is the case when B_i has a closed graph in $X \times X_i$, in particular, when $\text{cl}_{X_i} B_i$ is upper semicontinuous with closed values) for each $i \in I$ our definition of an equilibrium point for an abstract economy (generalized game) coincides with that of [4]. Also, according to Gale and Mas-Colell [5], a qualitative game is a family $\Gamma = (X_i, P_i)_{i \in I}$ of ordered pairs (X_i, P_i) where, for each $i \in I$, X_i is a topological space and $P_i : X = \prod_{j \in I} X_j \rightarrow 2^{X_i}$ is an irreflexive preference correspondence, i.e., $x_i \notin P_i(x)$ for all $x \in X$. A point $x^* \in X$ is said to be an equilibrium point of the qualitative game Γ if $P_i(x^*) = \emptyset$ for all $i \in I$.

In this paper we first prove one useful lemma. Next by applying an equilibrium existence theorem for a qualitative game due to Ding and Tan [4] and by employing an "approximation" method used by Tulcea [8], we obtain an equilibrium existence theorem for an abstract economy (generalized game). Our results generalize the corresponding results of Chang [3], Yannelis and Prab-

haker [10], Borglin and Keiding [2], Shafer and Sonnenschein [7], and Tulcea [8] in several ways.

2. PRELIMINARIES

We shall need the following simple result.

Lemma 2.1. *Let X be a topological space, Y a nonempty subset of a topological vector space E , $\mathcal{B}$ a base for the zero neighborhoods in E , and $B : X \rightarrow 2^Y$. For each $V \in \mathcal{B}$, let $B_V : X \rightarrow 2^Y$ be defined by $B_V(x) = (B(x) + V) \cap Y$ for each $x \in X$. If $\hat{x} \in X$ and $\hat{y} \in Y$ are such that $\hat{y} \in \bigcap_{V \in \mathcal{B}} \overline{B_V}(\hat{x})$, then $\hat{y} \in \overline{B}(\hat{x})$.*

Proof. Suppose $\hat{y} \notin \overline{B}(\hat{x})$. Then $(\hat{x}, \hat{y}) \notin \text{cl}_{X \times Y} \text{Graph}(B)$. Let U be an open neighborhood of $\hat{x}$ in X and $V \in \mathcal{B}$ be such that

$$(*) \quad (U \times (\hat{y} + V)) \cap \text{Graph}(B) = \emptyset.$$

Choose $W \in \mathcal{B}$ such that $W - W \subset V$. Since $\hat{y} \in \overline{B_W}(\hat{x})$, by assumption, $(\hat{x}, \hat{y}) \in \text{cl}_{X \times Y} \text{Graph}(B_W)$ so that $(U \times (\hat{y} + W)) \cap \text{Graph}(B_W) \neq \emptyset$. Take any $x \in U$ and $w_1 \in W$ with $(x, \hat{y} + w_1) \in \text{Graph}(B_W)$ so that $\hat{y} + w_1 \in B_W(x) = (B(x) + W) \cap Y$. Let $z \in B(x)$ and $w_2 \in W$ be such that $\hat{y} + w_1 = z + w_2 \in Y$. It follows that $z = \hat{y} + w_1 - w_2 \in \hat{y} + W - W \subset \hat{y} + V$, so $(\hat{y} + V) \cap B(x) \neq \emptyset$ where $x \in U$. This contradicts $(*)$. Thus we must have $\hat{y} \in \overline{B}(\hat{x})$. $\square$

We shall also need Theorem 1 and Theorem 3 of [4], which are stated below as Lemma 2.2 and Lemma 2.3 respectively.

Lemma 2.2. *Let X be a nonempty paracompact convex subset of a topological vector space and $P : X \rightarrow 2^X$ L_C -majorized. Suppose that there exist a nonempty compact convex subset X_0 of X and a nonempty compact subset K of X such that for each $y \in X \setminus K$ there is an $x \in \text{co}(X_0 \cup \{y\})$ with $x \in \text{co} P(y)$. Then there exists an $\hat{x} \in K$ such that $P(\hat{x}) = \emptyset$.*

Lemma 2.2 generalizes Theorem 5.1 and also Corollary 5.1 of Yannelis and Prabhakar [10] in the following ways:

- (i) The space X is paracompact instead of being compact.
- (ii) The correspondence P is $\mathcal{L}_C$ -majorized instead of being class $\mathcal{L}$ or $\mathcal{L}$ -majorized.

Lemma 2.3. *Let $\Gamma = (X_i, P_i)_{i \in I}$ be a qualitative game such that $X = \prod_{i \in I} X_i$ is paracompact. Suppose the following conditions are satisfied:*

- (i) *For each $i \in I$, X_i is a nonempty convex subset of a topological vector space.*
- (ii) *For each $i \in I$, $P_i : X \rightarrow 2^{X_i}$ is L_C -majorized.*
- (iii) $\bigcup_{i \in I} \{x \in X : P_i(x) \neq \emptyset\} = \bigcup_{i \in I} \text{int}_X \{x \in X : P_i(x) \neq \emptyset\}$.
- (iv) *There exist a nonempty compact convex subset X_0 of X and a nonempty compact subset K of X such that for each $y \in X \setminus K$ there is an $x \in \text{co}(X_0 \cup \{y\})$ with $x_i \in \text{co} P_i(y)$ for all $i \in I$.*

Then Γ has an equilibrium point in K .

3. EQUILIBRIUM POINTS

In this section we shall use the approximation technique used by Tulcea [8] to obtain equilibrium existence theorems for abstract economies. First we shall

prove the following existence theorem of “approximate” equilibrium points for a one-person game.

Theorem 3.1. *Let X be a nonempty paracompact convex subset of a topological vector space E and $A, B, P : X \rightarrow 2^X$ be such that*

- (i) *A is lower semicontinuous and, for each $x \in X$, $A(x)$ is nonempty and $\text{co } A(x) \subset B(x)$;*
- (ii) *$A \cap P$ is L_C -majorized;*
- (iii) *there exist a nonempty compact convex subset X_0 of X and a nonempty compact subset K of X such that, for each $y \in X \setminus K$,*

$$\text{co}(X_0 \cup \{y\}) \cap \text{co}(A(y) \cap P(y)) \neq \emptyset.$$

Then for each open convex neighborhood V of zero in E the one-person game $(X; A, \overline{B_V}; P)$ has an equilibrium point in K ; i.e., there exists a point $x_V \in K$ such that $x_V \in \overline{B_V}(x_V)$ and $A(x_V) \cap P(x_V) = \emptyset$ where $B_V(x) = (B(x) + V) \cap X$ for each $x \in X$.

Proof. Let V be an open convex neighborhood of zero in E , and define $A_V, B_V : X \rightarrow 2^X$ by $A_V(x) = (A(x) + V) \cap X$ and $B_V(x) = (B(x) + V) \cap X$ for each $x \in X$. Since A is lower semicontinuous, A_V has an open graph in $X \times X$ by Lemma 4.1 of [3, p. 244] or [8, p. 7]. By (i), $A_V(x) \subset B_V(x)$ for each $x \in X$. Let $F_V = \{x \in X : x \notin \overline{B_V}(x)\}$. Then F_V is open in X . Define $\Psi_V : X \rightarrow 2^X$ by

$$\Psi_V = \begin{cases} A(x) \cap P(x) & \text{if } x \notin F_V, \\ A_V(x) & \text{if } x \in F_V. \end{cases}$$

Suppose $x \in X$ is such that $\Psi_V(x) \neq \emptyset$.

Case 1: Suppose $x \in F_V$. Let $\Phi_x = A_V$ and $N_x = F_V$. Then N_x is an open neighborhood of x in X such that

- (a) for each $z \in N_x$, $\Psi_V(z) = A_V(z) = \Phi_x(z)$, $z \notin \overline{B_V}(z)$, so that $z \notin A_V(z) = \Phi_x(z)$,
- (b) for each $y \in X$, $\Phi_x^{-1}(y) = A_V^{-1}(y)$ is open in X since A_V has an open graph in $X \times X$.

Thus Φ_x is an L_C -majorant of Ψ_V at x .

Case 2: Suppose $x \notin F_V$. Then $\Psi_V(x) = A(x) \cap P(x) \neq \emptyset$. Since $A \cap P$ is L_C -majorized by (ii), there exist an open neighborhood N_x of x in X and a correspondence $\Phi'_x : X \rightarrow 2^X$ such that

- (a) $\Psi_V(z) = A(z) \cap P(z) \subset \Phi'_x(z)$ and $z \notin \text{co } \Phi'_x(z)$ for each $z \in N_x$,
- (b) for each $y \in X$, $(\Phi'_x)^{-1}(y)$ is compactly open in X .

Define $\Phi_x : X \rightarrow 2^X$ by

$$\Phi_x = \begin{cases} A_V(z) \cap \Phi'_x(z) & \text{if } z \notin F_V, \\ A_V(z) & \text{if } z \in F_V. \end{cases}$$

Note that

- (i) for each $z \in N_x$, clearly $\Psi_x^{-1}(y) \subset \Phi_x(z)$ and it is easy to see that $z \notin \text{co } \Phi_x(z)$,
- (ii) for each $y \in X$, $\Phi_x^{-1}(y) = [F_V \cup (\Phi'_x)^{-1}(y)] \cap A_V^{-1}(y)$ is compactly open in X .

Hence Φ_x is an L_C -majorant of Ψ_V at x .

Therefore, Ψ_V is an L_C -majorized correspondence. Moreover by (iii), for each $y \in X \setminus K$, there exists $x \in \text{co}(X_0 \cup \{y\}) \cap \text{co}(A(y) \cap P(y))$, so $x \in \text{co}(A(y) \cap P(y)) \subset \text{co} \Psi_V(y)$. Thus by Lemma 2.2 there exists $\hat{x} \in K$ such that $\Psi_V(\hat{x}) = \emptyset$. Since $A(\hat{x}) \neq \emptyset$ by (i), we must have $\hat{x} \in \overline{B_V}(\hat{x})$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$. $\square$

In the proof of Theorem 3.1, we obtained an equilibrium point of a one-person game as a maximal element of a binary relation. This idea is originated from Aliprantis and Brown [1].

The following is an existence of an "approximate" equilibrium point for an abstract economy.

Theorem 3.2. *Let I be any (finite or infinite) set. For each $i \in I$, let X_i be a nonempty convex subset of a topological vector space E_i and $A_i, B_i, P_i : X \rightarrow \prod_{j \in I} X_j \rightarrow 2^{X_i}$ be such that*

- (a) A_i is lower semicontinuous and, for each $x \in X$, $A_i(x)$ is nonempty and $\text{co} A_i(x) \subset B_i(x)$;
- (b) $A_i \cap P_i$ is L_C -majorized;
- (c) the set $E^i = \{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$ is open in X .

Suppose that X is paracompact and that there exist a nonempty compact convex subset X_0 of X and a nonempty compact subset K of X such that for each $y \in X \setminus K$ there is an $x \in \text{co}(X_0 \cup \{y\})$ with $x_i \in \text{co}(A_i(y) \cap P_i(y))$ for all $i \in I$. Then given any $V = \prod_{i \in I} V_i$ where, for each $i \in I$, V_i is an open convex neighborhood of zero in E_i , the abstract economy $\Gamma_V = (X_i; A_i, B_{V_i}; P_i)_{i \in I}$ has an equilibrium point in K ; i.e., there exists a point $x_V = (x_{V_i})_{i \in I} \in K$ such that $x_{V_i} \in \overline{B_{V_i}}(x_V)$ and $A_i(x_V) \cap P_i(x_V) = \emptyset$ for each $i \in I$ where $B_{V_i} = (B_i(x) + V_i) \cap X_i$ for each $x \in X$ and each $i \in I$.

Proof. Let $V = \prod_{i \in I} V_i$ be given where, for each $i \in I$, V_i is an open convex neighborhood of zero in E_i . Fix any $i \in I$, and define $A_{V_i}, B_{V_i} : X \rightarrow 2^{X_i}$ by $A_{V_i}(x) = (\text{co} A_i(x) + V_i) \cap X_i$ and $B_{V_i}(x) = (B_i(x) + V_i) \cap X_i$ for each $x \in X$. By (a), A_i is lower semicontinuous so that $\text{co} A_i$ is also lower semicontinuous by Proposition 2.6 of [6, p. 366]. It follows from Lemma 4.1 of [3, p. 244] or from [8, p. 7] that A_{V_i} has an open graph in $X \times X$. Now let $F_{V_i} = \{x \in X : x_i \notin \overline{B_{V_i}}(x)\}$. Then F_{V_i} is open in X . Define the map $Q_{V_i} : X \rightarrow 2^{X_i}$ by

$$Q_{V_i}(x) = \begin{cases} (A_i \cap P_i)(x) & \text{if } x \notin F_{V_i}, \\ A_i(x) & \text{if } x \in F_{V_i}. \end{cases}$$

We shall prove that the qualitative game $\mathcal{F} = (X_i, Q_{V_i})_{i \in I}$ satisfies all conditions of Lemma 2.3. First we note that for each $i \in I$ the set

$$\begin{aligned} \{x \in X : Q_{V_i}(x) \neq \emptyset\} &= F_{V_i} \cup \{x \in X \setminus F_{V_i} : A_i(x) \cap P_i(x) \neq \emptyset\} \\ &= F_{V_i} \cup ((X \setminus F_{V_i}) \cap E^i) = F_{V_i} \cup E^i \end{aligned}$$

is open in X by (c). Let $x \in X$ be such that $Q_{V_i}(x) \neq \emptyset$. We consider the following two cases.

Case 1: $x \in F_{V_i}$. Let $\Psi_x = A_{V_i}$ and $N_x = F_{V_i}$. Then N_x is an open neighborhood of x in X such that

- (i) $Q_{V_i}(z) \subset \Psi_x(z)$ and by (b) $z_i \notin \text{co} \Psi_x(z)$ for each $z \in N_x$;
- (ii) $\text{co} \Psi_x(z) \subset X_i$ for each $z \in X$ by (b);

- (iii) $\Psi_x^{-1}(y) = A_{V_i}^{-1}(y)$ is open in X for all $y \in X_i$ since A_{V_i} has an open graph.

Therefore, Ψ_x is an L_C -majorant of Q_{V_i} at x .

Case 2: $x \notin F_{V_i}$. Since $Q_{V_i}(x) = (A_i \cap P_i)(x) \neq \emptyset$ and $A_i \cap P_i$ is L_C -majorized, there exist an open neighborhood N_x of x in X and a correspondence $\phi_x : X \rightarrow 2^{X_i}$ such that

- (i) $(A_i \cap P_i)(z) \subset \phi_x(z)$ and $z_i \notin \text{co } \phi_x(z)$ for each $z \in N_x$;
- (ii) $\text{co } \phi_x(z) \subset X_i$ for each $z \in X$;
- (iii) $\phi_x^{-1}(y)$ is compactly open in X for each $y \in X_i$.

Define $\Psi_x : X \rightarrow 2^{X_i}$ by

$$\Psi_x(z) = \begin{cases} A_{V_i}(z) \cap \phi_x(z) & \text{if } z \notin F_{V_i}, \\ A_{V_i}(z) & \text{if } z \in F_{V_i}. \end{cases}$$

Note that as $(A_i \cap P_i)(z) \subset \phi_x(z)$ and $A_i(z) \subset A_{V_i}(z)$ for each $z \in N_x$, we have $Q_{V_i}(z) \subset \Psi_x(z)$ and $\text{co } \Psi_x(z) \subset X_i$. It is easy to see that $z_i \notin \text{co } \Psi_x(z)$ for all $z \in X$. Moreover, for any $y \in X_i$, the set

$$\begin{aligned} \Psi_x^{-1}(y) &= \{z \in X : y \in \Psi_x(z)\} \\ &= \{z \in X \setminus F_{V_i} : y \in \Psi_x(z)\} \cup \{z \in F_{V_i} : y \in \Psi_x(z)\} \\ &= \{z \in X \setminus F_{V_i} : y \in A_{V_i}(z) \cap \phi_x(z)\} \cup \{z \in F_{V_i} : y \in A_{V_i}(z)\} \\ &= [(X \setminus F_{V_i}) \cap A_{V_i}^{-1}(y) \cap \phi_x^{-1}(y)] \cup [F_{V_i} \cap A_{V_i}^{-1}(y)] \\ &= [\phi_x^{-1}(y) \cup F_{V_i}] \cap A_{V_i}^{-1}(y) \end{aligned}$$

is compactly open in X . Therefore, Ψ_x is an L_C -majorant of Q_{V_i} at point x .

Hence Q_{V_i} is an L_C -majorized correspondence. Now by assumption there exist a nonempty compact convex subset X_0 of X and a nonempty compact subset K of X such that for each $y \in X \setminus K$ there is an $x \in \text{co}(X_0 \cup \{y\})$ with $x_i \in \text{co}(A_i(y) \cap P_i(y))$ for all $i \in I$. Note that if $y \notin F_{V_i}$, then $x_i \in \text{co}(A_i(y) \cap P_i(y)) \subset \text{co } Q_{V_i}(y)$ and if $y \in F_{V_i}$, then $x_i \in \text{co}(A_i(y) \cap P_i(y)) \subset \text{co}(A_i(x)) = \text{co } Q_{V_i}(y)$. Thus, for each $i \in I$, $x_i \in \text{co } Q_{V_i}(y)$. Moreover, the set $\{x \in X : Q_{V_i}(x) \neq \emptyset\} = F_{V_i} \cup \{x \in X \setminus F_{V_i} : (A_i \cap P_i)(x) \neq \emptyset\} = F_{V_i} \cup E^i$ is open in X by (c). Therefore, all hypotheses of Lemma 2.3 are satisfied, so by Lemma 2.3 there exists a point $x_V = (x_{V_i})_{i \in I} \in K$ such that $Q_{V_i}(x_V) = \emptyset$ for all $i \in I$. Since, for each $i \in I$, $A_i(x)$ is nonempty, we must have $x_{V_i} \in \overline{B_{V_i}}(x_V)$ and $A_i(x_V) \cap P_i(x_V) = \emptyset$. $\square$

By Theorem 3.2 and Lemma 2.1, we shall present the main result of this paper as

Theorem 3.3. Let $\mathcal{G} = (X_i; A_i, B_i; P_i)_{i \in I}$ be an abstract economy such that $X = \prod_{i \in I} X_i$ is paracompact. Suppose the following conditions are satisfied:

- (a) For each $i \in I$, X_i is a nonempty convex subset of a locally convex Hausdorff topological vector space E_i .
- (b) For each $i \in I$, $A_i : X \rightarrow 2^{X_i}$ is lower semicontinuous such that, for each $x \in X$, $A_i(x)$ is nonempty and $\text{co } A_i(x) \subset B_i(x)$.
- (c) For each $i \in I$, $A_i \cap P_i$ is L_C -majorized.
- (d) For each $i \in I$, the set $E^i = \{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$ is open in X .

(e) *There exist a nonempty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$ there is an $x \in \text{co}(X_0 \cup \{y\})$ with $x_i \in \text{co}(A_i(y) \cap P_i(y))$ for all $i \in I$.*

Then $\mathcal{G}$ has an equilibrium point in K ; i.e., there exists a point $\hat{x} = (\hat{x}_i)_{i \in I} \in K$ such that, for each $i \in I$, $\hat{x}_i \in \overline{B_i}(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$.

Proof. For each $i \in I$, let $\mathcal{B}_i$ be the collection of all open convex neighborhoods of zero in E_i and $\mathcal{B} = \prod_{i \in I} \mathcal{B}_i$. Given any $V \in \mathcal{B}$, let $V = \prod_{j \in I} V_j$, where $V_j \in \mathcal{B}_j$ for each $j \in I$. By Theorem 3.2, there exists an $\hat{x}_V \in K$ such that $\hat{x}_{V_i} \in \overline{B_{V_i}}(\hat{x}_V)$ and $A_i(\hat{x}_V) \cap P_i(\hat{x}_V) = \emptyset$ for each $i \in I$, where $B_{V_i}(x) = (B_i(x) + V_i) \cap X_i$ for each $x \in X$. It follows that the set $Q_V := \{x \in K : x_i \in \overline{B_{V_i}}(x) \text{ and } A_i(x) \cap P_i(x) = \emptyset\}$ is a nonempty closed subset of K by (d).

Now we want to prove $\{Q_V\}_{V \in \mathcal{B}}$ has the finite intersection property. Let $\{V_1, \dots, V_n\}$ be any finite subset of $\mathcal{B}$. For each $i = 1, \dots, n$, let $V_i = \prod_{j \in I} V_{ij}$ where $V_{ij} \in \mathcal{B}_j$ for each $j \in I$; let $V = \prod_{j \in I} (\bigcap_{i=1}^n V_{ij})$. Then $Q_V \neq \emptyset$. Clearly $Q_V \subset \bigcap_{i=1}^n Q_{V_i}$ so that $\bigcap_{i=1}^n Q_{V_i} \neq \emptyset$. Therefore, the family $\{Q_V : V \in \mathcal{B}\}$ has the finite intersection property. Since K is compact, $\bigcap_{V \in \mathcal{B}} Q_V \neq \emptyset$. Now take any $\hat{x} \in \bigcap_{V \in \mathcal{B}} Q_V$. Then, for each $i \in I$, $\hat{x}_i \in \overline{B_{V_i}}(\hat{x})$ for each $V_i \in \mathcal{B}_i$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$. By Lemma 2.1, we also have, for each $i \in I$, $\hat{x}_i \in \overline{B_i}(\hat{x})$. $\square$

Corollary 3.4. *Let $\mathcal{G} = (X_i; A_i, B_i; P_i)_{i \in I}$ be an abstract economy such that $X = \prod_{i \in I} X_i$ is paracompact. Suppose the following conditions are satisfied:*

(a) *For each $i \in I$, X_i is a nonempty convex subset of a locally convex Hausdorff topological vector space E_i .*

(b) *For each $i \in I$ and for each $x \in X$, $A_i(x)$ is nonempty and $\text{co } A_i(x) \subset B_i(x)$.*

(c) *For each $i \in I$, A_i and P_i have open lower sections.*

(d) *For each $i \in I$, $A_i \cap P_i$ is L_C -majorized.*

(e) *There exist a nonempty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$ there is an $x \in \text{co}(X_0 \cup \{y\})$ with $x_i \in \text{co}(A_i(y) \cap P_i(y))$ for all $i \in I$.*

Then $\mathcal{G}$ has an equilibrium point in K ; i.e., there exists a point $\hat{x} \in K$ such that, for each $i \in I$, $\hat{x}_i \in \overline{B_i}(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$.

Proof. By (c), the map $A_i : X \rightarrow 2^{X_i}$ is lower semicontinuous and the set $E^i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is open in X . Therefore, all the hypotheses in Theorem 3.3 are satisfied, so the conclusion follows. $\square$

Corollary 3.4 (and hence also Theorem 3.3) improves Theorem 6.1 of Yan-nelis and Prabhakar [10] in the following ways:

- (i) the index set I need not be countable;
- (ii) for each $i \in I$, the set X_i need not be metrizable;
- (iii) for each $i \in I$, $A_i \cap P_i$ need not be L -majorized.

Corollary 3.5. *Let $\mathcal{G} = (X_i, A_i, B_i, P_i)_{i \in I}$ be an abstract economy such that $X = \prod_{i \in I} X_i$ is paracompact. Suppose the following conditions are satisfied:*

(a) *For each $i \in I$, X_i is a nonempty convex subset of a locally convex Hausdorff topological vector space.*

(b) For each $i \in I$ and for each $x \in X$, $A_i(x)$ is nonempty and $\text{co } A_i(x) \subset B_i(x)$.

(c) For each $i \in I$, A_i has an open graph in $X \times X_i$ (respectively, is lower semicontinuous) and P_i is lower semicontinuous (respectively, has an open graph in $X \times X_i$).

(d) For each $i \in I$, $A_i \cap P_i$ is L_C -majorized.

(e) There exist a nonempty compact convex subset X_0 of X and a nonempty compact subset K of X such that for each $y \in X \setminus K$ there is an $x \in \text{co}(X_0 \cup \{y\})$ with $x_i \in \text{co}(A_i(y) \cap P_i(y))$ for all $i \in I$.

Then $\mathcal{G}$ has an equilibrium point in K , i.e., there exists a point $\hat{x} \in X$ such that, for each $i \in I$, $\hat{x}_i \in \overline{B_i(\hat{x})}$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$.

Proof. For each $i \in I$, since A_i has an open graph in $X \times X_i$ (respectively, is lower semicontinuous) and P_i is lower semicontinuous (respectively, has open graph in $X \times X_i$), the map $A_i \cap P_i : X \rightarrow 2^{X_i}$ is also lower semicontinuous by Lemma 4.2 of [9, p. 103], so the set $E^i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is open in X . Therefore, all conditions of Theorem 3.3 are satisfied and the conclusion follows. $\square$

Corollary 3.5 (and hence also Theorem 3.3) generalizes Corollary 3 of Borglin and Keiding [2, p. 315], Theorem 4.1 of Chang [3, p. 247], Theorem of Shafer and Sonnenschein [7, p. 374], and Theorem 5 of Tulcea [8, p. 10].

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