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AN ADJOINT CHARACTERIZATION OF THE CATEGORY OF SETS

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ABSTRACT. If a category $\mathbf{B}$ with Yoneda embedding $Y: \mathbf{B} \to \mathbf{CAT}(\mathbf{B}^{op}, \mathbf{set})$ has an adjoint string, $U \dashv V \dashv W \dashv X \dashv Y$, then $\mathbf{B}$ is equivalent to $\mathbf{set}$.

1. INTRODUCTION

The statement of the abstract was implicitly conjectured in [9]. Here we establish the conjecture. We will see that it suffices to assume that $\mathbf{B}$ has an adjoint string $V \dashv W \dashv X \dashv Y$ with V pullback preserving.

A word on foundations and our notation is necessary. We write $\mathbf{set}$ for the category of small sets and assume that there is a Grothendieck topos, $\mathbf{SET}$, of sets which contains the set of arrows of $\mathbf{set}$ as an object. The 2-category of category objects in $\mathbf{SET}$, which we write $\mathbf{CAT}$, is cartesian closed, and $\mathbf{set}$ is an object of $\mathbf{CAT}$. Thus, for $\mathbf{C}$ a category in $\mathbf{CAT}$, $\mathbf{CAT}(\mathbf{C}^{op}, \mathbf{set})$ is also an object of $\mathbf{CAT}$, and we abbreviate it by $\mathscr{M}\mathbf{C}$ (it was written $\mathscr{P}\mathbf{C}$ in [8]). Substitution gives a 2-functor $\mathscr{M}: \mathbf{CAT}^{coop} \to \mathbf{CAT}$, where $\mathbf{CAT}^{coop}$ is the dual which reverses both arrows of $\mathbf{CAT}$ (functors) and 2-cells (natural transformations). A category $\mathbf{B}$ in $\mathbf{CAT}$ is said to be *locally small* if it has a hom functor $\mathbf{B}^{op} \times \mathbf{B} \to \mathbf{set}$ or equivalently a Yoneda embedding $Y = Y_{\mathbf{B}}: \mathbf{B} \to \mathscr{M}\mathbf{B}$. We say that a category $\mathbf{A}$ is *small* if the set of arrows of $\mathbf{A}$ is an object of $\mathbf{set}$. All categories under consideration, other than $\mathbf{SET}$ and $\mathbf{CAT}$, are objects of $\mathbf{CAT}$.

A functor $F: \mathbf{A} \to \mathbf{B}$ is said to be *Kan* if $\mathscr{M}F: \mathscr{M}\mathbf{B} \to \mathscr{M}\mathbf{A}$ has a left adjoint, denoted $\exists F$. if $\mathbf{A}$ is small and $\mathbf{B}$ is locally small, then F is Kan [8], but neither condition is necessary; if, say, we have $L \dashv F$, then $\mathscr{M}L \dashv \mathscr{M}F$ and $\exists F \cong \mathscr{M}L$. Smallness of $\mathbf{A}$ and local smallness of $\mathbf{B}$ also ensure that $\mathscr{M}F$ has a right adjoint, which we denote by $\forall F$. In particular, for small $\mathbf{A}$ the Yoneda embedding $Y_{\mathbf{A}}: \mathbf{A} \to \mathscr{M}\mathbf{A}$ yields $\exists(Y_{\mathbf{A}}) \dashv \mathscr{M}(Y_{\mathbf{A}}) \dashv \forall(Y_{\mathbf{A}}): \mathscr{M}\mathbf{A} \to \mathscr{M}\mathscr{M}\mathbf{A}$, and it is shown in [8] that $\forall(Y_{\mathbf{A}})$ is isomorphic to $Y_{\mathscr{M}\mathbf{A}}$. We can apply these considerations to $\mathbf{A} = \mathbf{0}$, the empty category, which is the initial object of $\mathbf{CAT}$. The unique functor $\mathbf{0} \to \mathscr{M}\mathbf{0} = \mathbf{1}$ is necessarily $Y_{\mathbf{0}}$ and gives rise to $\exists(Y_{\mathbf{0}}) \dashv \mathscr{M}(Y_{\mathbf{0}}) \dashv Y_{\mathbf{1}}: \mathbf{1} \to \mathscr{M}\mathbf{1}$. But $\mathscr{M}\mathbf{1}$ is isomorphic to $\mathbf{set}$ and $\mathbf{1}$ is terminal in $\mathbf{CAT}$, so the adjoint string is more conveniently labelled $\mathbf{0} \dashv ! \dashv \mathbf{1}: \mathbf{1} \to \mathbf{set}$.

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A further application of the result quoted from [8] gives an adjoint string of the kind mentioned in the abstract, namely,

$$\exists\mathbf{0} \dashv \mathscr{M}\mathbf{0} \dashv \mathscr{M}! \dashv \mathscr{M}\mathbf{1} \dashv Y_{\mathbf{set}}\colon \mathbf{set} \to \mathscr{M}\mathbf{set}.$$

We recall from [8] or [9] that a locally small category $\mathbf{B}$ is said to be *total* (abbreviating *totally cocomplete*) if $Y\colon \mathbf{B} \to \mathscr{M}\mathbf{B}$ has a left adjoint, X. Considerable motivation for the terminology is given in either reference. Examples include categories of algebras, categories of spaces, and categories of sheaves on a Grothendieck site. The reader is advised to keep in mind the situation when $\mathbf{B}$ is an ordered set and Y is replaced by its counterpart $\downarrow$ in the 2-category, **ord**, of ordered sets, order-preserving functions, and transformations. There $\downarrow\colon \mathbf{B} \to \mathscr{D}\mathbf{B}$ sends an element b to the down-closed subset of $\mathbf{B}$ consisting of all x such that $x \leq b$. ($\mathscr{D}\mathbf{B}$ is the lattice of all down-closed subsets of $\mathbf{B}$ ordered by inclusion.) This functor has a left adjoint, namely, supremum, $\bigvee$, precisely when $\mathbf{B}$ is (co)complete. It is helpful to think of X above as a generalization of $\bigvee$. Continuing the analogy, we recall from [1] that $\bigvee$ has a left adjoint precisely when $\mathbf{B}$ is (constructively) completely distributive. With this in mind we say that a total category is *totally distributive* when it has an adjoint string, $W \dashv X \dashv Y\colon \mathbf{B} \to \mathscr{M}\mathbf{B}$. The considerations in the previous paragraph show that $\mathscr{M}\mathbf{A}$ is totally distributive for small $\mathbf{A}$.

In the **ord** case a left adjoint for $\bigvee$ classifies the $\ll$, or "totally below", relation defined by $b \ll b'$ if and only if, for any D in $\mathscr{D}\mathbf{B}$, $b' \leq \bigvee D$ implies $b \in D$. A similar interpretation is possible for W. Its transpose, $\mathbf{B}^{\mathrm{op}} \times \mathbf{B} \to \mathrm{set}$, is in some respects like another hom functor. At least it makes good sense to think of its values as sets of "arrows", a priori distinct from the arrows of $\mathbf{B}$. A left adjoint, V, for W expresses a universal property with respect to the new arrows, and if this colimit-like functor itself has a left adjoint, then ordinary limits also distribute over these colimit-like universals.

The point of the heuristics of the preceding paragraph is that the adjoint strings we are considering are manifestations of "exactness". Given a suitably complete and cocomplete category $\mathbf{B}$, it seems possible, ab initio, that $\mathbf{B}$ is more distributive than **set**. The theorem of this paper shows that this is not the case. Exactness of a locally small category is strictly bounded by the exactness of **set**. Note further that while total categories $\mathbf{B}$ can fail to be cototal (that is, $\mathbf{B}^{\mathrm{op}}$ can fail to be total), totally distributive categories are always cototal. This and a detailed study of the heuristics above will appear in a separate forthcoming paper.

2. The adjoint characterization

Let $\mathbf{B}$ be a totally distributive category with adjoint string $W \dashv X \dashv Y\colon \mathbf{B} \to \mathscr{M}\mathbf{B}$. We write $\alpha, \beta\colon X \dashv Y$ to indicate that α is the unit and β is the counit for the adjunction. Since Y is fully faithful, β is an isomorphism and X is cofully faithful; i.e., $\mathbf{CAT}(X, \mathbf{C})$ is fully faithful for all $\mathbf{C}$. We write $\gamma, \delta\colon W \dashv X$ for the other adjunction. Cofully faithfulness of X implies that the unit, γ, is an isomorphism, so W is fully faithful. We define $\sigma\colon W \to Y$ to be the unique natural transformation satisfying $X\sigma \cdot \gamma = \beta^{-1}$. Equivalently, σ is the unique solution of $\beta \cdot X\sigma = \gamma^{-1}$. We write $I\colon \mathbf{E} \to \mathbf{B}$ for the *inverter* of $\sigma\colon W \to Y\colon \mathbf{B} \to \mathscr{M}\mathbf{B}$; i.e., $\mathbf{E}$ is the full subcategory of $\mathbf{B}$ determined by those B for which σ_B is an isomorphism. I is the resulting inclusion.

For any functor $F\colon \mathbf{C} \to \mathbf{D}$ with $\mathbf{D}(FC, D)$ in **set** for all C, D and for any $G\colon \mathbf{K} \to \mathbf{D}$, we follow Street and Walters [8] in writing $\mathbf{D}(F, G)\colon \mathbf{K} \to \mathscr{M}\mathbf{C}$ for the functor whose value at K in $\mathbf{K}$ is $\mathbf{D}(F-, GK)$. If $\mathbf{D}$ is locally small, $\mathbf{D}(F, G)$ is the composite

$$\mathbf{K} \xrightarrow{G} \mathbf{D} \xrightarrow{Y} \mathscr{M}\mathbf{D} \xrightarrow{\mathscr{M}F} \mathscr{M}\mathbf{C}.$$

Further, still assuming that $\mathbf{D}$ is locally small and for any $H\colon \mathbf{K} \to \mathscr{M}\mathbf{D}$, the Yoneda Lemma gives $\mathscr{M}\mathbf{D}(YF, H) \cong \mathscr{M}F \cdot H$ even though $\mathscr{M}\mathbf{D}$ need not be locally small.

Lemma 1. *A category* $\mathbf{B}$ *is equivalent to one of the form* $\mathscr{M}\mathbf{A}$ *with* $\mathbf{A}$ *small if and only if* $\mathbf{B}$ *is totally distributive and the inverter* I*, as above, is dense and Kan.*

Proof. (only if) We have already remarked that $\mathscr{M}\mathbf{A}$ is totally distributive for small $\mathbf{A}$. Here $\mathbf{E}$ is the Cauchy completion of $\mathbf{A}$. (Since this part of the lemma is not central to our present concerns we leave the proof of this claim as an exercise for the reader. In the **ord** case it is discussed in [5].) It is easy to see that I is dense and Kan.

(if) Given $\mathbf{B}$ and I as above, consider the composite

$$\mathbf{B} \xrightarrow{Y} \mathscr{M}\mathbf{B} \xrightarrow{\mathscr{M}I} \mathscr{M}\mathbf{E} = \mathbf{B}(I, 1_{\mathbf{B}}).$$

Since Y and $\mathscr{M}I$ have left adjoints, namely, X and $\exists I$ respectively, so does $\mathbf{B}(I, 1)$. We denote the left adjoint by $I * -$, since its value at Γ in $\mathscr{M}\mathbf{E}$, $I * \Gamma$, is the colimit of I weighted by Γ [8]. The unit for $I * - \dashv \mathbf{B}(I, 1)$ is an isomorphism since I is dense. The following isomorphisms are justified by (in order): definition of $I * -$, $W \dashv X$, σ is inverted by I, the Yoneda lemma, and fully faithfulness of $\exists I$ (which follows from fully faithfulness of I):

$$\begin{aligned} \mathbf{B}(I, I * \Gamma) &\cong \mathbf{B}(I, (X \cdot \exists I)(\Gamma)) \cong \mathscr{M}\mathbf{B}(WI, \exists I(\Gamma)) \\ &\cong \mathscr{M}\mathbf{B}(YI, \exists I(\Gamma)) \cong (\mathscr{M}I \cdot \exists I)(\Gamma) \cong \Gamma. \end{aligned}$$

Thus $\mathbf{B}(I, 1)\colon \mathbf{B} \to \mathscr{M}\mathbf{E}$ is an equivalence. Since both $\mathbf{E}$ and now $\mathscr{M}\mathbf{E}$ are locally small, it follows from [7] (see also [2]) that $\mathbf{E}$ is small as required. □

If $\mathbf{C}$ and $\mathbf{D}$ are total, then a functor $F\colon \mathbf{C} \to \mathbf{D}$ preserves all colimits if and only if it has a right adjoint. If, moreover, F is Kan, then preservation of all colimits is equivalent to invertibility of the canonical natural transformation $X_{\mathbf{D}}\exists F \to FX_{\mathbf{C}}$ as shown in the following left-hand diagram:

$$\begin{array}{ccc} \mathscr{M}\mathbf{C} & \xrightarrow{\exists F} & \mathscr{M}\mathbf{D} \\ {\scriptstyle X_{\mathbf{C}}}\downarrow & \cong & \downarrow{\scriptstyle X_{\mathbf{D}}} \\ \mathbf{C} & \xrightarrow[F]{} & \mathbf{D} \end{array} \qquad \begin{array}{ccc} \mathscr{M}\mathbf{C} & \xrightarrow{\exists F} & \mathscr{M}\mathscr{M}\mathbf{D} \\ {\scriptstyle X_{\mathbf{C}}}\downarrow & \cong & \downarrow{\scriptstyle \mathscr{M}Y_{\mathbf{D}}} \\ \mathbf{C} & \xrightarrow[F]{} & \mathscr{M}\mathbf{D} \end{array}$$

Again, the reader is advised to think of "X" as a general counterpart of the supremum arrow for a complete ordered set. Now replace $\mathbf{D}$ in the immediately preceding discussion by $\mathscr{M}\mathbf{D}$, where $\mathbf{D}$ is an arbitrary locally small category. According to our definition of total category and again invoking [7] (or [2]), $\mathscr{M}\mathbf{D}$ is total if and only if $\mathbf{D}$ is small. But we do have $\mathscr{M}Y_{\mathbf{D}}$ assuming only that $\mathbf{D}$ is locally small. If F is both Kan and a left adjoint, then a canonical isomorphism as in the above right-hand diagram is produced by a modification of the calculations which establish that the canonical arrow in the left-hand diagram is an isomorphism. Of course, we implicitly noted in the introduction that if $\mathbf{D}$ is small then $\mathscr{M}Y_{\mathbf{D}} \cong X_{\mathscr{M}\mathbf{D}}$. The point is that for $\mathbf{D}$ locally small $\mathscr{M}\mathbf{D}$ has the requisite weighted colimits, and they are provided by $\mathscr{M}Y_{\mathbf{D}}$.

Let $\mathbf{B}$ be a totally distributive category with $V \dashv W$. Then $W: \mathbf{B} \to \mathscr{M}\mathbf{B}$ is both Kan and a left adjoint. The considerations of the previous paragraph show that $WX \cong \mathscr{M}Y \cdot \exists W$. Since W is fully faithful, $XW \cong 1_{\mathbf{B}}$ and we have $\mathscr{M}Y \cdot \exists W \cdot W \cong W$. (This is a formulation for totally distributive categories of the "Interpolation Lemma" for constructively completely distributive lattices as in [5].) Now a calculation shows that the natural isomorphism above, $\mathscr{M}Y \cdot \exists W \cdot W \xrightarrow{\cong} W$, admits description by both

$$\mathscr{M}Y \cdot \exists W \cdot W \xrightarrow{\mathscr{M}Y \cdot \exists\sigma \cdot W} \mathscr{M}Y \cdot \exists Y \cdot W \cong W$$

and

$$\mathscr{M}Y \cdot \exists W \cdot W \xrightarrow{\mathscr{M}Y \cdot \exists W \cdot \sigma} \mathscr{M}Y \cdot \exists W \cdot Y \cong W \cdot X \cdot Y \cong W,$$

where both the first and last unnamed isomorphisms express the fully faithfulness of Y and the second unnamed isomorphism is an instance of $\mathscr{M}Y \cdot \exists W \cong WX$. These descriptions show that the profunctor $\mathbf{B} \to\!\!\!\!\!\!\!\!\;\;\mid\;\, \mathbf{B}$ determined by $W: \mathbf{B} \to \mathscr{M}\mathbf{B}$ carries an idempotent comonad structure, with counit determined by $\sigma: W \to Y$. It is convenient to define $T = VY: \mathbf{B} \to \mathbf{B}$. Then

$$\mathscr{M}Y \cdot \exists W \cdot \sigma \cong \mathscr{M}Y \cdot \mathscr{M}V \cdot \sigma \cong \mathscr{M}(VY) \cdot \sigma \cong \mathscr{M}T \cdot \sigma,$$

which shows that $\mathscr{M}T$ coinverts σ. By Lemma 4.3 of [4], T inverts σ.

Lemma 2. *A category* $\mathbf{B}$ *is equivalent to one of the form* $\mathscr{M}\mathbf{A}$ *with* $\mathbf{A}$ *a small, complete ordered set if and only if* $\mathbf{B}$ *is totally distributive with* $V \dashv W$.

Proof. (only if) A small, complete ordered set, $\mathbf{A}$, is a total category. Indeed, by definition $\downarrow_{\mathbf{A}}: \mathbf{A} \to \mathscr{D}\mathbf{A}$ has a left adjoint. So does the inclusion $\mathscr{D}\mathbf{A} \to \mathscr{M}\mathbf{A}$, and its composite with $\downarrow_{\mathbf{A}}$ is $Y: \mathbf{A} \to \mathscr{M}\mathbf{A}$, which therefore has a left adjoint. It follows that $\mathscr{M}\mathbf{A}$ has the required adjoint string.

(if) We saw above that $T = VY$ inverts $\sigma: W \to Y$. We denote the inverter $I: \mathbf{E} \to \mathbf{B}$ as above, so there exists a unique functor $H: \mathbf{B} \to \mathbf{E}$ such that $IH = T$. We show $H \dashv I$ by showing that $\mathbf{E}(H, 1) \cong \mathbf{B}(1, I)$. Now

$$\begin{aligned}\mathbf{B}(1, I) &\cong YI \cong WI \cong \mathscr{M}\mathbf{B}(Y, WI) \cong \mathbf{B}(VY, I)\\ &\cong \mathbf{B}(T, I) \cong \mathbf{B}(IH, I) \cong \mathbf{E}(H, 1),\end{aligned}$$

where we have the last isomorphism because I is fully faithful. From $H \dashv I$ we have I Kan (with $\exists I \cong \mathscr{M}H$). To see that I is dense consider

$$\begin{aligned}I * - \cdot \mathbf{B}(I, 1) &\cong X \cdot \exists I \cdot \mathscr{M}I \cdot Y \cong X \cdot \mathscr{M}H \cdot \mathscr{M}I \cdot Y = X \cdot \mathscr{M}(IH) \cdot Y\\ &= X \cdot \mathscr{M}(T) \cdot Y \cong X \cdot \mathbf{B}(T, 1) = X \cdot \mathbf{B}(VY, 1)\\ &\cong X \cdot \mathscr{M}\mathbf{B}(Y, W) \cong X \cdot W \cong 1_{\mathbf{B}}.\end{aligned}$$

By (the proof of) Lemma 1, $\mathbf{B}$ is equivalent to $\mathscr{M}\mathbf{E}$ and the equivalence $\mathbf{B}(I, 1)$ identifies I and $Y_{\mathbf{E}}$. Thus $H \dashv I$ shows that $\mathbf{E}$ is total (directly, although that was already clear above since a full reflective subcategory of a total is total) and hence complete in the usual sense. But from Lemma 1 we also have $\mathbf{E}$ small so, by [3, Exercise 3D], $\mathbf{E}$ is an ordered set. □

Theorem 3. *A category* **B** *is equivalent to* **set** *if and only if* **B** *is totally distributive with* $V \dashv W$ *and* V *preserves pullbacks.*

Proof. (only if) This follows from the introduction, for if we have $U \dashv V$ then certainly V preserves pullbacks.

(if) Now $T = VY$ preserves pullbacks. It follows from the construction of H in Lemma 2 that H preserves pullbacks, so **E** is "lex total", meaning that the defining left adjoint for totality is left exact. (It necessarily preserves the terminal object.) By [6] **E** is a Grothendieck topos (for since **E** is small the size requirement in [6] is trivially satisfied). But since by Lemma 2, **E** is also an ordered set, it must therefore be **1**. Indeed, we have **true** = **false**: $1 \to \Omega$ in **E**. □

Corollary 4. *The category* **set** *is characterized by* $U \dashv V \dashv W \dashv X \dashv Y$.

References

1. B. Fawcett and R. J. Wood, *Constructive complete distributivity.* I, Math. Proc. Cambridge Philos. Soc. **107** (1990), 81–89.
2. F. Foltz, *Légitimité des catégories de préfaisceaux*, Diagrammes **1** (1979), 1–5.
3. P. J. Freyd, *Abelian categories*, Harper and Row, New York, 1964.
4. R. Paré, R. Rosebrugh, and R. J. Wood, *Idempotents in bicategories*, Bull. Austral. Math. Soc. **39** (1989), 421–434.
5. R. Rosebrugh and R. J. Wood, *Constructive complete distributivity.* IV, Applied Categorical Structures (to appear).
6. R. Street, *Notions of topos*, Bull. Austral. Math. Soc. **23** (1981), 199–208.
7. ——, unpublished manuscript, 1979.
8. R. Street and R. F. C. Walters, *Yoneda structures on 2-categories*, J. Algebra **50** (1978), 350–379.
9. R. J. Wood, *Some remarks on total categories*, J. Algebra **75** (1982), 538–545.

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