VALUES OF L -FUNCTIONS AT THE CRITICAL POINT

LIEM MAI

(Communicated by Dennis A. Hejhal)

ABSTRACT. For a discriminant D of a binary quadratic form, we study the average value of $L(s, \varepsilon_D)$ at the critical point $\frac{1}{2}$ where ε_D is defined by W. Kohnen and D. Zagier :

$$\varepsilon_D(n) = \sum_{\substack{g>0 \\ g|\delta, g^2|n \\ (\delta/g, n/g^2)=1}} \left(\frac{D_0}{g^{-2}n} \right) g$$

for $n \in \mathbb{N}$ and $D = D_0\delta^2$, D_0 a fundamental discriminant and $\delta \in \mathbb{N}$. When $D = D_0$, $L(s, \varepsilon_{D_0})$ is the Dirichlet series $L(s, (\frac{D_0}{\cdot}))$. We derive an asymptotic formula for $\sum_D L(\frac{1}{2}, \varepsilon_D)$, where the sum runs over all discriminants $D \in (0, Y]$ or $[-Y, 0)$.

Let D be any discriminant of a binary quadratic form, i.e., $D \in \mathbb{Z}$ and $D \equiv 0, 1 \pmod{4}$. The full modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})/\pm I$ operates on the set of binary quadratic forms $f(u, v)$ by

$$(\gamma \circ f)(u, v) = f(au + cv, bu + dv)$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

In [10], Zagier defined

$$\zeta(s, D) = \sum_f \sum_{\substack{(m, n) \in \mathbb{Z}^2 / \mathrm{Aut}(f) \\ f(m, n) > 0}} \frac{1}{f(m, n)^s}$$

where the first sum runs over all Γ equivalence classes of binary quadratic forms $f(u, v)$ of discriminant D and the second over all inequivalent pairs of integers with respect to the group of units $\mathrm{Aut}(f) = \{\gamma \in \Gamma : \gamma f = f\}$. He proved that

$$\zeta(s, D) = \zeta(2s) \sum_{n=1}^{\infty} a_n n^{-s}$$

Received by the editors February 3, 1993.

1991 *Mathematics Subject Classification.* Primary 11C54.

Key words and phrases. L -functions, discriminants.

where a_n is the number of solutions $x \pmod{2n}$ of the congruence $x^2 \equiv D \pmod{4n}$ and $\zeta(s, D)$ can be expressed in terms of standard Dirichlet series:

$$\begin{aligned}\zeta(s, D) &= \zeta(s) L_D(s) \\ &= \zeta(s) L_{D_0}(s) \sum_{d|\delta} \mu(d) \left(\frac{D_0}{d}\right) d^{-s} \sigma_{1-2s}\left(\frac{\delta}{d}\right)\end{aligned}$$

where $D = D_0 \delta^2$, D_0 is a fundamental discriminant, $\left(\frac{D_0}{d}\right)$ is the Kronecker symbol, $L_{D_0}(s)$ is the Dirichlet series attached to $\left(\frac{D_0}{n}\right)$, and $\sigma_k(m) = \sum_{d|m} d^k$.

In [5], Kohnen and Zagier defined, for $D = D_0 \delta^2$:

$$\varepsilon_D(n) = \sum_{\substack{g>0 \\ g|\delta, g^2|n \\ (\delta/g, n/g^2)=1}} \left(\frac{D_0}{g^{-2}n}\right) g;$$

then

$$L_D(s) = \sum_{n=1}^{\infty} \varepsilon_D(n) n^{-s} = L(s, \varepsilon_D) \quad (\text{say}).$$

Note that ε_D is slightly different from $\psi_D(\cdot) = \left(\frac{D}{\cdot}\right)$. If $D = D_0$, a fundamental discriminant, then $\varepsilon_D(n) = \psi_{D_0}(n) = \left(\frac{D_0}{n}\right)$ and $L(s, \varepsilon_D)$ turns out to be the Dirichlet series $L_{D_0}(s) = L(s, \psi_{D_0}) = L(s, \left(\frac{D_0}{\cdot}\right))$; and if $D = D_0 \delta^2$ where D_0 is a fundamental discriminant, it equals the product of $L_{D_0}(s)$ with a finite Euler product over the divisors of δ .

For ψ_D , using the theory of quadratic forms, Siegel (see [8]) proved that

$$\sum_{0 < D \leq Y} L(1, \psi_D) = CY + O(Y^{1/2} \log Y)$$

and Vinogradov and Tahtadzjan (see [9]), by using exponential sums, proved that

$$\sum_{0 < D \leq Y} L\left(\frac{1}{2}, \psi_D\right) = Y\{C \log Y + C'\} + O(Y^{1/2+\varepsilon})$$

where C and C' are explicit constants.

For fundamental discriminants D_0 , using a mean square estimate for quadratic character sums, Jutila (see [4]) proved a conjecture by Goldfeld and Viola (see [3]) which states that

$$\sum_{\substack{0 < \pm D_0 \leq Y \\ D_0 \text{ fund. disc.}}} L_{D_0}\left(\frac{1}{2}\right) = Y\{C_1 \log Y + C'_1\} + O(Y^{3/4+\varepsilon})$$

and later, using Eisenstein series of half integral weight, Goldfeld and Hoffstein (see [2]) improved the error term:

$$\sum_{\substack{0 < \pm D_0 \leq Y \\ D_0 \text{ fund. disc.}}} L_{D_0}\left(\frac{1}{2}\right) = Y\{C_1 \log Y + C'_1\} + O(Y^{19/32+\varepsilon})$$

where C, C' are explicit constants.

In this paper, we will prove a similar result for ε_D , where D is a discriminant, using the nice functional equation satisfied by $L(s, \varepsilon_D)$:

$$L\left(\frac{1}{2} + s, \varepsilon_D\right) = D^s \pi^s \frac{\Gamma\left(\frac{1}{4} - \frac{s}{2}\right)}{\Gamma\left(\frac{1}{4} + \frac{s}{2}\right)} L\left(\frac{1}{2} - s, \varepsilon_D\right)$$

as they are Fourier coefficients of Eisenstein series (see [10], for example). The method is far simpler than the previous ones, and the error term is better than the one obtained by Jutila.

Theorem 1.

$$\sum_{\substack{0 < D \leq Y \\ D \equiv 0, 1 \pmod{4} \\ t^2 | D}} L\left(\frac{1}{2}, \varepsilon_D\right) = C_1(t)Y \log Y + C_2(t)Y + O(Y^{2/3} \log Y),$$

where

$$C_1(t) = \begin{cases} \frac{5}{16t^2} \sum_{t_1|t} \frac{\phi(t_1)}{t_1} & \text{if } t \text{ is odd,} \\ \frac{1}{2t^2} \sum_{t_1|t} \frac{\phi(t_1)}{t_1} & \text{if } t \text{ is even} \end{cases}$$

and

$$C_2(t) = \begin{cases} \frac{5}{16t^2} \sum_{t_1|t} \frac{\phi(t_1)}{t_1} \left(4\gamma - \log \pi - 1 - \frac{2 \log 4}{5} \right. \\ \quad \left. + \frac{\Gamma'}{\Gamma}\left(\frac{1}{4}\right) - 4 \log\left(\frac{t}{t_1}\right) + 4 \sum_{p|t_1} \frac{\log p}{p-1} \right) & \text{if } t \text{ is odd,} \\ \frac{1}{2t^2} \sum_{t_1|t} \frac{\phi(t_1)}{t_1} \left(+4\gamma - \log \pi - 1 \right. \\ \quad \left. + \frac{\Gamma'}{\Gamma}\left(\frac{1}{4}\right) - 4 \log\left(\frac{t}{t_1}\right) + 4 \sum_{p|t_1} \frac{\log p}{p-1} \right) & \text{if } t \text{ is even.} \end{cases}$$

Proof. We begin with the integral

$$\int_{(\sigma)} L\left(\frac{1}{2} + s, \varepsilon_D\right) X^s \Gamma(s) ds = \sum_{n=1}^{\infty} \varepsilon_D(n) \frac{1}{\sqrt{n}} e^{-n/X}$$

where $\Re(s) = \sigma > 1$.

Moving the line of integration to the left to a line $\Re(s) = -\eta$, we get

$$L\left(\frac{1}{2}, \varepsilon_D\right) + \frac{1}{2\pi i} \int_{(-\eta)} L\left(\frac{1}{2} + s, \varepsilon_D\right) X^s \Gamma(s) ds.$$

Summing over all discriminants D , $0 < D \leq Y$, $t^2 | D$, we have

$$\begin{aligned} \sum_{\substack{0 < D \leq Y \\ D \equiv 0, 1 \pmod{4} \\ t^2 | D}} L\left(\frac{1}{2}, \varepsilon_D\right) &= \sum_{\substack{0 < D \leq Y \\ D \equiv 0, 1 \pmod{4} \\ t^2 | D}} \sum_{n=1}^{\infty} \varepsilon_D(n) \frac{1}{\sqrt{n}} e^{-n/X} \\ &\quad - \frac{1}{2\pi i} \int_{(-\eta)} \sum_{\substack{0 < D \leq Y \\ D \equiv 0, 1 \pmod{4} \\ t^2 | D}} L\left(\frac{1}{2} + s, \varepsilon_D\right) X^s \Gamma(s) ds \\ &= S + I. \end{aligned}$$

At first, we assume that $(t, 2) = 1$.

For the sum S , we write $S = S_{\text{main}} + S_{\text{error}}$, where the first sum is over all squares and the latter over all nonsquares.

For the sum S_{main} , we have

$$\begin{aligned}
 S_{\text{main}} &= \sum_{m=1}^{\infty} \frac{1}{m} e^{-m^2/X} \sum_{\substack{0 < D \leq Y \\ D \equiv 0, 1 \pmod{4} \\ i^2 | D}} \varepsilon_D(m^2) \\
 &= \sum_{m=1}^{\infty} \frac{1}{m} e^{-m^2/X} \sum_{g|m} g \sum_{\substack{0 < D \leq Y, D = \delta^2 D_0 \\ D_0 \text{ fund. disc.} \\ g|\delta, (\delta/g, m/g)=1 \\ i^2 | D}} \left(\frac{D_0}{m^2 g^{-2}} \right) \\
 &= \sum_{m=1}^{\infty} \frac{1}{m} e^{-m^2/X} \sum_{g|m} g \sum_{\substack{0 < D \leq Y \\ D = \delta^2 D_0 \\ D_0 \text{ fund. disc.} \\ g|\delta, i^2 | D}} \left(\frac{D_0}{m^2 g^{-2}} \right) \left(\frac{\delta^2 g^{-2}}{m^2 g^{-2}} \right) \\
 &= \sum_{m=1}^{\infty} \frac{1}{m} e^{-m^2/X} \sum_{g|m} g \left(\sum_{\substack{0 < D_1 < Y/g^2 \\ g^2 D_1 \equiv 0, 1 \pmod{4} \\ g^2 D_1 \equiv 0 \pmod{i^2}}} \left(\frac{D_1}{m^2 g^{-2}} \right) \right).
 \end{aligned}$$

For each $g|m$, let $(g, t) = t_1$; then $g = g_1 t_1$, $(g_1, t) = 1$, and $t = t_1 t_2$. We have

$$S_{\text{main}} = \sum_{m=1}^{\infty} \frac{1}{m} e^{-m^2/X} \sum_{\substack{g|m \\ (m/g, t_2)=1}} g \sum_{\substack{0 < D_1 < Y/g^2 \\ g^2 D_1 \equiv 0, 1 \pmod{4} \\ g^2 D_1 \equiv 0 \pmod{i_2^2}}} \left(\frac{D_1}{m^2 g^{-2}} \right).$$

If g is even, the innermost sum is

$$\sum_{\substack{0 < D_1 < Y/g^2 \\ g^2 D_1 \equiv 0 \pmod{i_2^2} \\ (D_1, m/g)=1}} 1 = \frac{\phi(mg^{-1})}{mg^{-1}} \frac{Y}{g^2 t_2^2} + O(\mathbf{d}(mg^{-1})),$$

and if g is odd, it is

$$\sum_{\substack{0 < D_1 < Y/g^2 \\ D_1 \equiv 0, 1 \pmod{4} \\ g^2 D_1 \equiv 0 \pmod{i_2^2} \\ (D_1, m/g)=1}} 1 = \frac{1}{2} \frac{\phi(mg^{-1})}{mg^{-1}} \frac{Y}{g^2 t_2^2} + O(\mathbf{d}(mg^{-1}))$$

where $\mathbf{d}(n) = \#$ of divisors of n . Recall that g is odd if and only if g_1 is

odd, as $g = g_1 t_1$ and t_1 is odd. Therefore ,

$$S_{\text{main}} = \sum_{m=1}^{\infty} \frac{1}{m} e^{-m^2/X} \left\{ \sum_{\substack{g|m \\ (\frac{m}{g}, t_2)=1 \\ g \text{ is even}}} \frac{\phi(mg^{-1})}{mg^{-1}} \frac{Y}{gt_2^2} + \sum_{\substack{g|m \\ (\frac{m}{g}, t_2)=1 \\ g \text{ is odd}}} \frac{\phi(mg^{-1})}{mg^{-1}} \frac{1}{2} \frac{Y}{gt_2^2} \right\} \\ + \sum_{m=1}^{\infty} \frac{1}{m} e^{-m^2/X} \left(O \left(\sum_{g|m} g \mathbf{d}(mg^{-1}) \right) \right).$$

Note that the condition $(m/g, t_2) = (m/g_1 t_1, t_2) = 1$ implies that $m = m_1 t_1$, where $(m_1, t_2) = 1$. Therefore, we can write the sum S_{main} as

$$\sum_{t_1|t} \sum_{\substack{m_1=1 \\ (m_1, t_2)=1}}^{\infty} \frac{1}{m_1 t_1} e^{-m_1^2 t_1^2/X} \\ \times \left\{ \sum_{\substack{g_1|m_1 \\ (\frac{m_1}{g_1}, t_2)=1 \\ g_1 \text{ is even}}} \frac{\phi(m_1 g_1^{-1})}{m_1 g_1^{-1}} \frac{Y}{g_1 t_1 t_2^2} + \sum_{\substack{g_1|m_1 \\ (\frac{m_1}{g_1}, t_2)=1 \\ g_1 \text{ is odd}}} \frac{\phi(m_1 g_1^{-1})}{m_1 g_1^{-1}} \frac{1}{2} \frac{Y}{g_1 t_1 t_2^2} \right\} \\ + \sum_{t_1|t} \sum_{\substack{m_1=1 \\ (m_1, t_2)=1}}^{\infty} \frac{1}{m_1 t_1} e^{-m_1^2 t_1^2/X} O \left(\sum_{g_1|m_1} g_1 t_1 \mathbf{d} \left(\frac{m_1}{g_1} \right) \right) \\ = \sum_{t_1|t} \left\{ \sum_{\substack{g_1 \\ (g_1, t_2)=1 \\ g_1 \text{ is even}}} \sum_{\substack{h_1=1 \\ (h_1, t_2)=1}}^{\infty} \frac{1}{g_1^2} \frac{\phi(h_1)}{h_1^2} \frac{Y}{t^2} e^{-g_1^2 h_1^2 t_1^2/X} \right. \\ \left. + \sum_{\substack{g_1 \\ (g_1, t_2)=1 \\ g_1 \text{ is odd}}} \sum_{\substack{h_1=1 \\ (h_1, t_2)=1}}^{\infty} \frac{1}{2} \frac{1}{g_1^2} \frac{\phi(h_1)}{h_1^2} \frac{Y}{t^2} e^{-g_1^2 h_1^2 t_1^2/X} \right\} \\ + O \left(\sum_{t_1|t} \sum_{g_1} \sum_{h_1} \frac{1}{h_1} \mathbf{d}(h_1) e^{-g_1^2 h_1^2 t_1^2/X} \right).$$

The last term is $O(\mathbf{d}(t) X^{1/2} (\log X)^2)$.

The first sum is

$$\frac{1}{4} \frac{Y}{t^2} \sum_{t_1|t} \sum_{\substack{g_2=1 \\ (g_2, t_2)=1}}^{\infty} \frac{1}{g_2^2} \sum_{\substack{h_1=1 \\ (h_1, t_2)=1}}^{\infty} \frac{\phi(h_1)}{h_1^2} e^{-g_2^2 h_1^2/(X/4t_1^2)} \\ = \frac{1}{4} \frac{Y}{t^2} \sum_{t_1|t} \int_{(\sigma)} \zeta(2s+1) \prod_{p|t_2} \left(1 - \frac{1}{p^{2s+1}} \right) \left(\frac{X}{4t_1^2} \right)^s \Gamma(s) ds.$$

We move the line of integration to the left to the line $\Re(s) = -1 + \frac{1}{\log X}$. Using the Laurent expansions at $s = 0$, we have:

$$\begin{aligned}\zeta(1+2s) &= \frac{1}{2s} + \gamma + \cdots, \\ \left(\frac{X}{4t_1^2}\right)^s &= 1 + \log\left(\frac{X}{4t_1^2}\right)s + \cdots, \\ \Gamma(s) &= \frac{1}{s} - \gamma + \cdots,\end{aligned}$$

$$\prod_{p|t_2} \left(1 - \frac{1}{p^{2s+1}}\right) = \frac{\phi(t_2)}{t_2} + \frac{\phi(t_2)}{t_2} \left(\sum_{p|t_2} \frac{2\log p}{p-1}\right)s + \cdots$$

where γ is the Euler constant. Then the first sum is

$$\begin{aligned}& \frac{1}{4} \frac{Y}{t^2} \sum_{t_1|t} \frac{\phi(t_2)}{t_2} \left(\frac{1}{2}(\log X) + \left(\frac{\gamma}{2} - \frac{1}{2} \log(4t_1^2) + \sum_{p|t_2} \frac{\log p}{p-1} \right) \right) \\ & + O\left(\frac{Y}{X} \sum_{t_1|t} \frac{1}{t_1}\right) + O(\mathbf{d}(t)X^{1/2}(\log X)^2) \\ & = \frac{1}{4} \frac{Y}{t^2} \sum_{t_1|t} \frac{\phi(t_1)}{t_1} \left(\frac{1}{2}(\log X) + \left(\frac{\gamma}{2} - \frac{1}{2} \log(4t_2^2) + \sum_{p|t_1} \frac{\log p}{p-1} \right) \right) \\ & + O\left(\frac{Y}{X} \sum_{t_1|t} \frac{1}{t_1}\right) + O(\mathbf{d}(t)X^{1/2}(\log X)^2)\end{aligned}$$

as $t = t_1 t_2$, t_1 and t_2 are interchangeable.

The second term is

$$\begin{aligned}& \frac{1}{2} \sum_{t_1|t} \left(\sum_{\substack{s_1=1 \\ (s_1, t_2)=1}}^{\infty} \sum_{h_1} - \sum_{\substack{s_1=1 \\ (s_1, t_2)=1 \\ s_1 \text{ is even}}}^{\infty} \sum_{h_1} \right) \\ & = \frac{Y}{t^2} \sum_{t_1|t} \frac{\phi(t_1)}{t_1} \left(\frac{3}{16}(\log X) + \frac{3}{16}\gamma + \frac{\log 4}{16} - \frac{3}{8}(\log t_2) + \frac{3}{8} \sum_{p|t_1} \frac{\log p}{p-1} \right) \\ & + O\left(\frac{Y}{X} \sum_{t_1|t} \frac{1}{t_1}\right) + O(\mathbf{d}(t)X^{1/2}(\log X)^2).\end{aligned}$$

Therefore,

$$\begin{aligned}S_{\text{main}} &= \frac{Y}{t^2} \sum_{t_1|t} \frac{\phi(t_1)}{t_1} \left(\frac{5}{16}(\log X) + \frac{5}{16}\gamma - \frac{\log 4}{16} - \frac{5}{8}(\log t_2) + \frac{5}{8} \sum_{p|t_1} \frac{\log p}{p-1} \right) \\ & + O\left(\frac{Y}{X} \sum_{t_1|t} \frac{1}{t_1}\right) + O(\mathbf{d}(t)X^{1/2}(\log X)^2).\end{aligned}$$

Next, for the sum S_{error} , we have

$$S_{\text{error}} = \sum_{\substack{n=1 \\ n \text{ is nonsquare}}}^{\infty} \frac{1}{\sqrt{n}} e^{-n/X} \left(\sum_{\substack{0 < D \leq Y \\ D \equiv 0, 1 \pmod{4} \\ i^2 | D}} \varepsilon_D(n) \right).$$

We need a lemma:

Lemma. *For a nonsquare integer n , we have*

$$\sum_{\substack{0 < D \leq Y \\ D \equiv 0, 1 \pmod{4} \\ i^2 | D}} \varepsilon_D(n) = O(\sqrt{n}(\log n)\gamma(n))$$

where $\gamma(n) = \sum_{g^2 | n} 1$.

Proof.

$$\begin{aligned} \sum_{\substack{0 < D \leq Y \\ D \equiv 0, 1 \pmod{4} \\ i^2 | D}} \varepsilon_D(n) &= \sum_{\substack{0 < D \leq Y \\ D \equiv 0, 1 \pmod{4} \\ i^2 | D}} \sum_{\substack{g \\ g^2 | n, g | \delta \\ \text{for } D = \delta^2 D_0 \\ (\delta | g, n/g^2) = 1}} \left(\frac{D_0}{n g^{-2}} \right) g \\ &= \sum_{g^2 | n} g \left(\sum_{\substack{0 < D \leq Y \\ g^2 | D \\ D \equiv 0, 1 \pmod{4} \\ i^2 | D}} \left(\frac{D_0}{n g^{-2}} \right) \left(\frac{\delta^2 g^{-2}}{n g^{-2}} \right) \right) \\ &= \sum_{g^2 | n} g \left(\sum_{\substack{0 < D_1 \leq Y/g^2 \\ g^2 D_1 \equiv 0, 1 \pmod{4} \\ g^2 D_1 \equiv 0 \pmod{i^2}}} \left(\frac{D_1}{n g^{-2}} \right) \right) = O \left(\sum_{g^2 | n} g \frac{\sqrt{n}}{g} \log n \right) \end{aligned}$$

by the Polya-Vinogradov inequality.

Hence,

$$\sum_{\substack{0 < D \leq Y \\ D \equiv 0, 1 \pmod{4} \\ i^2 | D}} \varepsilon_D(n) = O(\sqrt{n}(\log n)\gamma(n)).$$

Returning to our sum S_{error} , we have

$$\begin{aligned} S_{\text{error}} &= O \left(\sum_{\substack{n=1 \\ n \text{ is nonsquare}}}^{\infty} \frac{1}{\sqrt{n}} e^{-n/X} \sqrt{n}(\log n)\gamma(n) \right) \\ &= O \left(\sum_{n \leq X} (\log n)\gamma(n) \right) = O \left(\log X \sum_{n \leq X} \sum_{g^2 | n} 1 \right) \\ &= O \left(\log X \sum_{g \leq \sqrt{X}} \sum_{h \leq X/g^2} 1 \right) = O(X \log X). \end{aligned}$$

Now for the integral I , we have

$$I = -\frac{1}{2\pi i} \int_{(-\eta)} \sum_{\substack{0 < D \leq Y \\ D \equiv 0, 1 \pmod{4} \\ i^2 | D}} L\left(\frac{1}{2} + s, \varepsilon_D\right) X^s \Gamma(s) ds$$

where $L(s, \varepsilon_D)$ satisfies a functional equation

$$L\left(\frac{1}{2} + s, \varepsilon_D\right) = D^s \pi^s \frac{\Gamma(\frac{1}{4} - \frac{s}{2})}{\Gamma(\frac{1}{4} + \frac{s}{2})} L\left(\frac{1}{2} - s, \varepsilon_D\right).$$

Substituting into our integral I , we have

$$\begin{aligned} I &= -\frac{1}{2\pi i} \int_{(-\eta)} \sum_{\substack{0 < D \leq Y \\ D \equiv 0, 1 \pmod{4} \\ i^2 | D}} L\left(\frac{1}{2} - s, \varepsilon_D\right) D^{-s} \pi^s \frac{\Gamma(\frac{1}{4} - \frac{s}{2})}{\Gamma(\frac{1}{4} + \frac{s}{2})} X^s \Gamma(s) ds \\ &= -\frac{1}{2\pi i} \int_{(-\eta)} \sum_{\substack{0 < D \leq Y \\ D \equiv 0, 1 \pmod{4} \\ i^2 | D}} \sum_{n=1}^{\infty} \frac{\varepsilon_D(n)}{n^{1/2-s}} D^{-s} \pi^s X^s \frac{\Gamma(\frac{1}{4} - \frac{s}{2})}{\Gamma(\frac{1}{4} + \frac{s}{2})} \Gamma(s) ds \\ &= I_{\text{main}} + I_{\text{error}}. \end{aligned}$$

For the integral I_{main} which corresponds to squares $n = m^2$, we have

$$\sum_{\substack{0 < D \leq Y \\ D \equiv 0, 1 \pmod{4} \\ i^2 | D}} \varepsilon_D(m^2) D^{-s} = \sum_{g|m} g^{1-2s} \left(\sum_{\substack{0 < D_1 < Y/g^2 \\ g^2 D_1 \equiv 0, 1 \pmod{4} \\ g^2 D_1 \equiv 0 \pmod{i^2}}} \chi_0^{(mg^{-1})}(D_1) D_1^{-s} \right)$$

where $\chi_0^{(mg^{-1})}(\cdot)$ is the principal character modulo mg^{-1} .

For each g , let $(g, t) = t_1$; then $g = g_1 t_1$, $t = t_1 t_2$, and the above sum is

$$\begin{aligned} &= \sum_{t_1 | t} \sum_{\substack{g|m \\ (m/g, t_2)=1}} g^{1-2s} \left(\sum_{\substack{0 < D_1 < Y/g^2 \\ g^2 D_1 \equiv 0, 1 \pmod{4} \\ g^2 D_1 \equiv 0 \pmod{i^2}}} \chi_0^{(mg^{-1})}(D_1) D_1^{-s} \right) \\ &= \sum_{t_1 | t} \sum_{\substack{g|m \\ (m/g, t_2)=1}} g^{1-2s} \left(\sum_{\substack{0 < D_1 < Y/g^2 \\ g^2 D_1 \equiv 0, 1 \pmod{4} \\ g_1^2 D_1 \equiv 0 \pmod{i_2^2}}} \chi_0^{(mg^{-1})}(D_1) D_1^{-s} \right). \end{aligned}$$

The innermost sum for g even, by partial summation, is

$$\frac{\phi(mg^{-1})}{mg^{-1}} \left(\frac{Y}{g^2 t_2^2} \right)^{1-s} \frac{t_2^{-2s}}{1-s} + O \left(Y^\eta g^{-2\eta} \mathbf{d} \left(\frac{m}{g} \right) (|s| + 1) \right)$$

on $\Re(s) = -\eta$. For g is odd, it is

$$\frac{1}{2} \frac{\phi(mg^{-1})}{mg^{-1}} \left(\frac{Y}{g^2 t_2^2} \right)^{1-s} \frac{t_2^{-2s}}{1-s} + O \left(Y^\eta g^{-2\eta} \mathbf{d} \left(\frac{m}{g} \right) (|s| + 1) \right).$$

Then

$$\begin{aligned} & \sum_{\substack{0 < D \leq Y \\ D \equiv 0, 1 \pmod{4} \\ t_1^2 | D}} \sum_{m=1}^{\infty} \frac{\varepsilon_D(m^2)}{m^{1-2s}} D^{-s} \\ &= \sum_{t_1 | t} \sum_{\substack{m_1=1 \\ (m_1, t_2)=1}}^{\infty} \left\{ \sum_{\substack{g_1 | m_1 \\ (\frac{m_1}{g_1}, t_2)=1 \\ g_1 \text{ is even}}} \left(\frac{1}{m_1 t_1} \right)^{1-2s} (g_1 t_1)^{1-2s} \left(\frac{Y}{g_1^2 t^2} \right)^{1-s} \frac{\phi(m_1 g_1^{-1})}{m_1 g_1^{-1}} \frac{t_2^{-2s}}{1-s} \right. \\ & \quad \left. + \sum_{\substack{g_1 | m_1 \\ (\frac{m_1}{g_1}, t_2)=1 \\ g_1 \text{ is odd}}} \left(\frac{1}{m_1 t_1} \right)^{1-2s} (g_1 t_1)^{1-2s} \left(\frac{Y}{g_1^2 t^2} \right)^{1-s} \frac{1}{2} \frac{\phi(m_1 g_1^{-1})}{m_1 g_1^{-1}} \frac{t_2^{-2s}}{1-s} \right\} \\ & \quad + O \left(\sum_{t_1 | t} \sum_{\substack{g_1 \\ (g_1, t_2)=1}} \sum_{\substack{h_1=1 \\ (h_1, t_2)=1}}^{\infty} \frac{g_1 t_1}{(g_1 h_1 t_1)^{1+2\eta}} Y^\eta \mathbf{d}(h_1) (|s| + 1) \right) \\ &= \sum_{t_1 | t} \left\{ \sum_{\substack{g_1 \\ (g_1, t_2)=1 \\ g_1 \text{ is even}}} \sum_{\substack{h_1=1 \\ (h_1, t_2)=1}}^{\infty} \frac{1}{h_1^{1-2s}} \frac{Y^{1-s}}{1-s} \left(\frac{1}{g_1 t} \right)^{2-2s} \frac{\phi(h_1)}{h_1} \left(\frac{t_1}{t} \right)^{2s} \right. \\ & \quad \left. + \sum_{\substack{g_1 \\ (g_1, t_2)=1 \\ g_1 \text{ is odd}}} \sum_{\substack{h_1=1 \\ (h_1, t_2)=1}}^{\infty} \frac{1}{h_1^{1-2s}} \frac{Y^{1-s}}{1-s} \left(\frac{1}{g_1 t} \right)^{2-2s} \frac{\phi(h_1)}{h_1} \left(\frac{t_1}{t} \right)^{2s} \right\} \\ & \quad + O \left(Y^\eta \left(\sum_{t_1 | t} \right) (|s| + 1) \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{t^2} \sum_{t_1|t} t_1^{2s} \left\{ \sum_{\substack{g_1 \\ (g_1, t_2)=1 \\ g_1 \text{ is even}}} \left(\frac{1}{g_1}\right)^{2-2s} \sum_{\substack{h_1=1 \\ (h_1, t_2)=1}}^{\infty} \frac{\phi(h_1)}{h_1^{2-2s}} \frac{Y^{1-s}}{1-s} \right. \\
&\quad \left. + \frac{1}{2} \sum_{\substack{g_1 \\ (g_1, t_2)=1 \\ g_1 \text{ is odd}}} \left(\frac{1}{g_1}\right)^{2-2s} \sum_{\substack{h_1=1 \\ (h_1, t_2)=1}}^{\infty} \frac{\phi(h_1)}{h_1^{2-2s}} \frac{Y^{1-s}}{1-s} \right\} \\
&\quad + O\left(Y^\eta \left(\sum_{t_1|t} \frac{1}{t_1}\right) (|s|+1)\right) \\
&= \frac{1}{t^2} \sum_{t_1|t} t_1^{2s} \left(\frac{1}{2} + \frac{1}{2} \left(\frac{1}{2}\right)^{2-2s}\right) \zeta(1-2s) \prod_{p|t_2} (1-p^{-1+2s}) \frac{Y^{1-s}}{1-s} \\
&\quad + O\left(Y^{1/2} \left(\sum_{t_1|t} \frac{1}{t_1}\right) (|s|+1)\right)
\end{aligned}$$

if we choose $\eta = \frac{1}{2} + \frac{1}{\log Y}$.

Therefore,

$$\begin{aligned}
I_{\text{main}} &= -\frac{1}{2\pi i} \frac{1}{2t^2} \left\{ \int_{(-\eta)} \sum_{t_1|t} t_1^{2s} \prod_{p|t_2} (1-p^{-1+2s}) \frac{Y^{1-s}}{1-s} \left(1 + \left(\frac{1}{2}\right)^{2-2s}\right) \zeta(1-2s) \right. \\
&\quad \left. \times (\pi X)^s \frac{\Gamma(\frac{1}{4} - \frac{s}{2})}{\Gamma(\frac{1}{4} + \frac{s}{2})} \Gamma(s) ds \right\} \\
&\quad + O\left(Y^{1/2} X^{1/2} \left(\sum_{t_1|t} \frac{1}{t_1}\right)\right) \\
&= -\frac{Y}{2t^2} \frac{1}{2\pi i} \left\{ \int_{(-\eta)} \sum_{t_1|t} t_1^{2s} \prod_{p|t_2} (1-p^{-1+2s}) \frac{1}{1-s} \left(1 + \left(\frac{1}{2}\right)^{2-2s}\right) \zeta(1-2s) \right. \\
&\quad \left. \times \left(\frac{\pi X}{Y}\right)^s \frac{\Gamma(\frac{1}{4} - \frac{s}{2})}{\Gamma(\frac{1}{4} + \frac{s}{2})} \Gamma(s) ds \right\} \\
&\quad + O\left(Y^{1/2} X^{1/2} \frac{1}{t^2} \left(\sum_{t_1|t} \frac{1}{t_1}\right)\right).
\end{aligned}$$

Recall that $t = t_1 t_2$. Now use the Laurent expansions at $s = 0$:

$$\begin{aligned}
\zeta(1-2s) &= -\frac{1}{2s} + \gamma + \cdots, \\
\Gamma(s) &= \frac{1}{s} - \gamma + \cdots,
\end{aligned}$$

$$\frac{1}{1-s} \left(\frac{\pi X}{Y} \right)^s \frac{\Gamma(\frac{1}{4} - \frac{s}{2})}{\Gamma(\frac{1}{4} + \frac{s}{2})} = 1 + \left(1 - \frac{\Gamma'}{\Gamma} \left(\frac{1}{4} \right) + \log \pi + \log \left(\frac{X}{Y} \right) \right) s + \dots,$$

$$1 + \left(\frac{1}{2} \right)^{2-2s} = \frac{5}{4} + \frac{\log 4}{4} s + \dots,$$

$$\sum_{t_1|t} t_1^{2s} \prod_{p|t_2} (1 - p^{-1+2s}) = \sum_{t_1|t} \frac{\phi(t_2)}{t_2} + \sum_{t_1|t} \frac{\phi(t_2)}{t_2} \left(2 \log t_1 - \sum_{p|t_2} \frac{2 \log p}{p-1} \right) s + \dots.$$

Moving the line of integration from $\Re(s) = -\eta$ to the right to the line

$$\Re(s) = \frac{1}{2} - \frac{1}{\log(Y/X)}$$

and picking up the residue at $s = 0$, we have

$$I_{\text{main}} = \frac{Y}{t^2} \left(\sum_{t_1|t} \frac{\phi(t_1)}{t_1} \right) \left(-\frac{5}{16} \log \left(\frac{X}{Y} \right) - \frac{5}{16} \log \pi + \frac{15}{16} \gamma - \frac{5}{16} + \frac{5}{16} \frac{\Gamma'}{\Gamma} \left(\frac{1}{4} \right) \right. \\ \left. - \frac{\log 4}{16} - \frac{5}{8} (\log t_2) + \frac{5}{8} \sum_{p|t_1} \frac{\log p}{p-1} \right) + O \left(Y^{1/2} X^{1/2} \frac{1}{t^2} \sum_{t_1|t} t_1 \right).$$

Now for the error term I_{error} , which corresponds to nonsquares n , we have

$$I_{\text{error}} = -\frac{1}{2\pi i} \int_{(-\eta)} \sum_{\substack{n=1 \\ n \text{ nonsquare}}}^{\infty} \frac{1}{n^{1/2-s}} \sum_{\substack{0 < D \leq Y \\ D \equiv 0, 1 \pmod{4} \\ t^2 | D}} \varepsilon_D(n) D^{-s} X^s \pi^s \frac{\Gamma(\frac{1}{4} - \frac{s}{2})}{\Gamma(\frac{1}{4} + \frac{s}{2})} \Gamma(s) ds \\ = -\frac{1}{2\pi i} \int_{(-\eta)} \sum_{\substack{n \leq U \\ n \text{ nonsquare}}} + \frac{1}{2\pi i} \int_{(-\eta)} \sum_{\substack{n > U \\ n \text{ nonsquare}}} = I_{\text{error 1}} + I_{\text{error 2}}$$

where $U = Y^4/X^2$.

For the first integral $I_{\text{error 1}}$, using Polya-Vinogradov inequality and partial summation, we have

$$\left| \sum_{\substack{0 < D \leq Y \\ D \equiv 0, 1 \pmod{4} \\ t^2 | D}} \varepsilon_D(n) D^{-s} \right| = O(\sqrt{n} (\log n) \gamma(n) Y^\eta).$$

Then

$$I_{\text{error 1}} = O \left(Y^\eta X^{-\eta} \sum_{\substack{n \leq U \\ n \text{ nonsquare}}} \frac{(\log n) \gamma(n)}{n^\eta} \right)$$

by the exponential decay of the Gamma function.

But

$$\begin{aligned} \sum_{\substack{n \leq U \\ n \text{ nonsquare}}} \frac{(\log n) \gamma(n)}{n^\eta} &= \sum_{\substack{n \leq U \\ n \text{ nonsquare}}} \frac{\log n}{n^\eta} \sum_{g^2 | n} 1 \\ &= \sum_{\substack{g \leq \sqrt{U} \\ n \text{ nonsquare}}} \frac{1}{g^{2\eta}} \sum_{n \leq U/g^2} \frac{\log n}{n^\eta} = O((\log U) U^{1-\eta}), \end{aligned}$$

then

$$I_{\text{error } 1} = \left(\left(\frac{Y}{X} \right)^\eta U^{1-\eta} (\log U) \right).$$

If we choose $\eta = 1 - \frac{1}{\log(Y^3/X)}$, then

$$I_{\text{error } 1} = O \left(\left(\frac{Y}{X} \right)^\eta \frac{Y^{4-4\eta}}{X^{2-2\eta}} (\log Y) \right) = O \left(\frac{Y}{X} (\log Y) \right).$$

Next, for the integral $I_{\text{error } 2}$, using the trivial bound, we have

$$\begin{aligned} I_{\text{error } 2} &= O \left(\sum_{\substack{n > U \\ n \text{ nonsquare}}} \frac{1}{n^{1/2+\eta}} Y^{1+\eta} X^{-\eta} \right) \\ &= O \left(Y^{1+\eta} X^{-\eta} U^{1/2-\eta} \right) = O \left(\frac{Y^3}{X} \left(\frac{X}{Y^3} \right)^\eta \right) = O(1). \end{aligned}$$

Therefore

$$I_{\text{error}} = O \left(\frac{Y}{X} (\log Y) \right).$$

In summary, we have

$$\begin{aligned} \sum_{\substack{0 < D \leq Y \\ D \equiv 0, 1 \pmod{4} \\ t_1^2 | D}} L \left(\frac{1}{2}, \varepsilon_D \right) &= \frac{Y}{t^2} \sum_{t_1 | t} \frac{\phi(t_1)}{t_1} \left\{ \frac{5}{16} \log Y + \frac{5}{4} \gamma - \frac{5}{16} \log \pi - \frac{\log 4}{8} - \frac{5}{16} \right. \\ &\quad \left. + \frac{5}{16} \frac{\Gamma'}{\Gamma} \left(\frac{1}{4} \right) - \frac{5}{4} \log \left(\frac{t}{t_1} \right) + \frac{5}{4} \sum_{p | t_1} \frac{\log p}{p-1} \right\} \\ &\quad + O \left(Y^{1/2} X^{1/2} \frac{1}{t^2} \left(\sum_{t_1 | t} t_1 \right) \right) + O \left(\frac{Y}{X} (\log Y) \right) \\ &\quad + O \left(\frac{Y}{X} \sum_{t_1 | t} \frac{1}{t_1} \right) + O \left(\mathbf{d}(t) X^{1/2} (\log X)^2 \right) + O(X \log X). \end{aligned}$$

The error term is $O(Y^{2/3}(\log Y))$ if we choose $X = Y^{1/3}$.

In the case that $(t, 2) \neq 1$, we have

$$\sum_{\substack{0 < D \leq Y \\ D \equiv 0, 1 \pmod{4} \\ t^2 | D}} L\left(\frac{1}{2}, \varepsilon_D\right) = \frac{Y}{t^2} \sum_{t_1 | t} \frac{\phi(t_1)}{t_1} \left\{ \frac{1}{2} \log Y + 2\gamma - \frac{\log \pi}{2} - \frac{1}{2} + \frac{1}{2} \frac{\Gamma'}{\Gamma}\left(\frac{1}{4}\right) \right. \\ \left. - 2 \log\left(\frac{t}{t_1}\right) + 2 \sum_{p | t_1} \frac{\log p}{p-1} \right\} + O\left(Y^{2/3}(\log Y)\right).$$

This concludes the proof of the theorem.

Remark. For small values of t , we used the Polya-Vinogradov inequality to get upper bounds for character sums. For large values of t , other inequalities can be used, and we can get the error term in Theorem 1 divided by a function of t , but the exponent of Y can be larger.

Taking $t = 1$, we have

Corollary.

$$\sum_{\substack{0 < D \leq Y \\ D \equiv 0, 1 \pmod{4}}} L\left(\frac{1}{2}, \varepsilon_D\right) = \frac{5}{16} Y \log Y + \frac{5}{16} Y \left(4\gamma - \log \pi - \frac{2 \log 4}{5} - 1 + \frac{\Gamma'}{\Gamma}\left(\frac{1}{4}\right)\right) \\ + O(Y^{2/3} \log Y).$$

For the case $D \in [-Y, 0)$, the corresponding functional equation is

$$L\left(\frac{1}{2} + s, \varepsilon_D\right) = |D|^s \pi^s \frac{\Gamma(\frac{3}{4} - \frac{s}{2})}{\Gamma(\frac{3}{4} + \frac{s}{2})} L\left(\frac{1}{2} - s, \varepsilon_D\right);$$

then we have

Theorem 2.

$$\sum_{\substack{-Y < D < 0 \\ D \equiv 0, 1 \pmod{4} \\ t^2 | D}} L\left(\frac{1}{2}, \varepsilon_D\right) = \begin{cases} \frac{5}{16} \frac{Y}{t^2} \sum_{t_1 | t} \frac{\phi(t_1)}{t_1} \left(\log Y + 4\gamma - \log \pi - \frac{2 \log 4}{5} - 1 + \frac{\Gamma'}{\Gamma}\left(\frac{3}{4}\right) \right. \\ \quad \left. - 4 \log\left(\frac{t}{t_1}\right) + 4 \sum_{p | t_1} \frac{\log p}{p-1} \right) + O(Y^{2/3}(\log Y)) \\ \quad \text{if } t \text{ is odd,} \\ \frac{1}{2} \frac{Y}{t^2} \sum_{t_1 | t} \frac{\phi(t_1)}{t_1} \left(\log Y + 4\gamma - \log \pi - 1 + \frac{\Gamma'}{\Gamma}\left(\frac{3}{4}\right) \right. \\ \quad \left. - 4 \log\left(\frac{t}{t_1}\right) + 4 \sum_{p | t_1} \frac{\log p}{p-1} \right) + O(Y^{2/3}(\log Y)) \\ \quad \text{if } t \text{ is even.} \end{cases}$$

ACKNOWLEDGMENTS

The author would like to thank Professor Ram Murty for his suggestions and discussions during this work. He also thanks the referee for various helpful comments.

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THE FIELDS INSTITUTE FOR RESEARCH IN MATHEMATICAL SCIENCES, 185 COLUMBIA STREET
WEST, WATERLOO, ONTARIO, CANADA N2L 5Z5
E-mail address: mai@fields.uwaterloo.ca