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SPECIAL POINTS IN COMPACT SPACES

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ABSTRACT. Given a collection $\mathscr{C}$, of cardinality κ, of subsets of a compact space X, we prove the existence of a point x such that whenever $C \in \mathscr{C}$ and $X \in \overline{C}$, there exists a G_λ-set Z with $\lambda < \kappa$ and $x \in Z \subset \overline{C}$. We investigate the case when $\mathscr{C}$ is the collection of all cozerosets of X and also when X is a dyadic space. We apply this result to homogeneous compact spaces. Another application is a characterization of 2^{ω_1} among dyadic spaces.

1. INTRODUCTION

A space X is *homogeneous* if for every $x, y \in X$, there exists an autohomeomorphism h of X such that $h(x) = y$. This paper is concerned with homogeneous compact spaces. For a cardinal κ and a collection $\mathscr{C}$ of subsets of a space X, we define what we mean by a $\mathscr{C}_\kappa$-point of X (see Definition 2.1). Our main results are the Point Theorem and the Dyadic Cozero Point Theorem of §2 which prove the existence of $\mathscr{C}_\kappa$-points in compact spaces. In §3 we apply the results of §2 to homogeneous spaces and get the Homogeneous Theorem and the Dyadic Cozero Homogeneous Theorem. This paper contains a number of examples illustrating the sharpness of the theorems and also answering two questions previously posed by the author. Finally, we characterize 2^{ω_1}, among dyadic spaces, as the unique zero-dimensional, homogeneous compact space of weight ω_1. This answers a question posed by B. Efimov for the case $\kappa = \omega_1$ of 2^κ. We first proved this result assuming the continuum hypothesis CH. Subsequently, L. Shapiro removed this extra assumption. We show how it follows from our general theory.

Cardinals are initial ordinals, and ω and ω_1 are the first two infinite cardinals. The cofinality of the cardinal κ is denoted by $\mathrm{cf}(\kappa)$. The set of all subsets of X is denoted by $\mathscr{P}(X)$. We use standard set-theory notions as in Kunen's Set Theory [Ku].

All spaces in this paper are assumed to be T_2, i.e., Hausdorff. We use standard topological notions as in Engleking's General Topology [En2] with one exception. We use the term cozeroset instead of the term functionally open. We emphasize two cardinal functions. The weight $w(X)$ of a space X is the

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least cardinal of a base for X, and the character $\chi(x, X)$ of x in X is the least cardinal of a local base at x in X. We emphasize one topological property. A space X is *dyadic* if X is the continuous image of some Cantor Cube 2^κ.

2. The point results

We begin with the central definitions of our paper. Let κ be an infinite cardinal. A subset Z of a space X is a G_κ-*set* (resp. $G_{<\kappa}$-set) if $Z = \bigcap_{\alpha<\lambda} U_\alpha$ where $\lambda \leq \kappa$ (resp. $\lambda < \kappa$) and the U_α's are open subsets of X. For $A \subset X$, we put $G_\kappa\text{-int}(A) = \bigcup\{Z \subset A : Z \text{ is a } G_\kappa\text{-set of } X\}$. We say that A is G_κ-open in X if $A = G_\kappa\text{-int}(A)$. The preceding two notions yield two more if we uniformly replace κ by $<\kappa$.

Definition 2.1. Let κ be an infinite cardinal, and let $\mathscr{C} \subset \mathscr{P}(X)$. A point $x \in X$ is a $\mathscr{C}_\kappa$-point of X if whenever $C \in \mathscr{C}$ and $x \in \overline{C}$, then $x \in G_{<\kappa}\text{-int}(\overline{C})$.

Definition 2.2. Let κ be an infinite cardinal. A point $x \in X$ is a W_κ-point of X if there exists a disjoint clopen family $\mathscr{B}$ in X of cardinality κ such that every neighbourhood N of x intersects all but finitely many members of $\mathscr{B}$.

When the compact space X is understood, we put $\mathscr{COZ} = \{C \subset X : C \text{ is a cozeroset of } X\}$. Note that in a compact space X a cozeroset is the same as an open F_σ set.

Proposition 2.3. *Let X be a compact space, and let κ be an infinite cardinal. Then, a W_κ-point is not a $\mathscr{COZ}_\kappa$-point.*

Proof. Let $\mathscr{B}$ be a disjoint, clopen family of cardinality κ witnessing the fact that x is a W_κ-point of X. Let $\mathscr{C}$ be a countably infinite subfamily of $\mathscr{B}$. Then $C = \bigcup \mathscr{C}$ is a cozeroset of X and $x \in \overline{C}$.

Let $\lambda < \kappa$, and let Z be any G_λ-set with $x \in Z$. Using regularity, for each $\alpha < \lambda$, get a closed neighbourhood N_α of x with $\bigcap_{\alpha<\lambda} N_\alpha \subset Z$. Using the W_κ-point property of x, choose $B \in \mathscr{B} \backslash \mathscr{C}$ such that for every finite $F \subset \lambda$, $B \cap \bigcap_{\alpha \in F} N_\alpha \neq \varnothing$. Using compactness of X, choose $p \in B \cap \bigcap_{\alpha<\lambda} N_\alpha$. Note that $B \cap C = \varnothing$. Thus, $p \in Z$ and $p \notin \overline{C}$. Hence x cannot be a $\mathscr{COZ}_\kappa$-point of X. □

Theorem 2.4 (Point Theorem). *If X is a compact space and $\mathscr{C}$ is a collection of subsets of X with $|\mathscr{C}| = \kappa$, then there exists a $\mathscr{C}_\kappa$-point in X.*

Proof. Enumerate $\mathscr{C}$ as $\{C_\alpha : \alpha < \kappa\}$. We will define, by induction on $\alpha < \kappa$, a sequence of zerosets Z_α of X. Put $Z_0 = X$. At stage α if there exists a zeroset Z such that $Z \cap \bigcap_{\beta<\alpha} Z_\beta \neq \varnothing$ and $Z \cap \overline{C_\alpha} = \varnothing$, then let Z_α be one such zeroset Z. If there does not exist such a zeroset Z, then put $Z_\alpha = X$.

At the end of this induction, by compactness, choose $x \in \bigcap_{\alpha<\kappa} Z_\alpha$. Then x is a $\mathscr{C}_\kappa$-point of X. To see this, let $C \in \mathscr{C}$ and assume that $x \in \overline{C}$. Choose $\alpha < \kappa$ with $C = C_\alpha$. We claim that $\bigcap_{\beta \leq \alpha} Z_\beta \subset \overline{C_\alpha}$. If not, then choose $y \in \bigcap_{\beta \leq \alpha} Z_\beta \backslash \overline{C_\alpha}$. Get a zeroset Z with $y \in Z$ and $Z \cap \overline{C_\alpha} = \varnothing$. So at stage α we have that $Z_\alpha \cap \overline{C_\alpha} = \varnothing$. But this is a contradiction. □

The Point Theorem only has content if each $x \in X$ has $\chi(x, X) \geq \kappa$, because a point of character $<\kappa$ is obviously a $\mathscr{C}_\kappa$-point.

We are mostly interested when $\mathscr{C} = \mathscr{COZ}$. Since $|\mathscr{COZ}| = w(X)^\omega$ for compact spaces X, to avoid no content, we restate this special case as:

Theorem 2.5 (Cozero Point Theorem). *If X is compact and $w(X) = \kappa = \kappa^\omega$, then there exists a $\mathscr{COZ}_\kappa$-point in X.*

We now give some examples to illustrate the sharpness, in general, of $\mathscr{COZ}_\kappa$ in the Cozero Point Theorem. Obviously, if $\lambda < \kappa$, then having a $\mathscr{COZ}_\lambda$-point is a stronger result than having a $\mathscr{COZ}_\kappa$-point. In certain spaces there are better results. Every point is a $\mathscr{COZ}_{\omega_1}$-point in any Cantor cube 2^κ. Every point is a $\mathscr{COZ}_\omega$-point in any compact basically disconnected space. In the following we will use the basic fact that in a compact space X a W_κ-point is not a $\mathscr{COZ}_\kappa$-point.

Example 2.6. For every $\kappa > \omega$ there exists a compact X_κ such that

- $w(X) = \kappa$;
- X has no $\mathscr{COZ}_\lambda$-points, for $\lambda < \kappa$.

Proof. *Case* 1. κ is a successor; $\kappa = \lambda^+$ where $\lambda \geq \omega$. Put $\alpha\lambda = \lambda \cup \{\infty\}$, the Alexandroff one-point compactification of the discrete space λ. For $f \in (\alpha\lambda)^\kappa$, define $\operatorname{supp}(f) = \{\beta < \kappa : f(\beta) \neq \infty\}$. Set $X_\kappa = \{f \in (\alpha\lambda)^\kappa : f|\operatorname{supp}(f)$ is a one-to-one function$\}$. As a subspace of $(\alpha\lambda)^\kappa$, X_κ is closed and hence is compact. Let $f \in X_\kappa$. Choose $\beta \in \kappa \setminus \operatorname{supp}(f)$. For $\gamma < \lambda$, define $B_\gamma = \{g \in X_\kappa : g(\beta) = \gamma\}$. Then $\{B_\gamma : \gamma < \lambda\}$ witnesses the fact that f is a W_λ-point. Hence, all points of X_κ are W_λ-points, and therefore no point is a $\mathscr{COZ}_\lambda$-point.

We mention that X_{ω_2} answers a question of the author raised in [Be1] about whether there exists a compact space in which every point is a W_{ω_1}-point.

Case 2. κ is a limit; $\kappa = \sum_{\alpha<\operatorname{cf}(\kappa)} \kappa_\alpha$ where for each $\alpha < \operatorname{cf}(\kappa)$, κ_α is a successor cardinal $< \kappa$. Set $X_\kappa = \prod_{\alpha<\operatorname{cf}(\kappa)} X_{\kappa_\alpha}$ where the X_{κ_α}'s are as in Case 1. Fix $\lambda < \kappa$, and assume that $f \in X_\kappa$ is a $\mathscr{COZ}_\lambda$-point. Choose $\alpha < \operatorname{cf}(\kappa)$ such that $\lambda < \kappa_\alpha < \kappa$. Then $f(\alpha)$ must be a $\mathscr{COZ}_\lambda$-point of X_{κ_α}—a contradiction. □

The next example shows that the condition $\kappa = \kappa^\omega$ cannot be removed, in general, from the Cozero Point Theorem.

Example 2.7. There exists a compact space Y of weight ω_ω which has no $\mathscr{COZ}_{\omega_\omega}$-points.

Proof. For each $n < \omega$, define $Y_n = X_{\omega_{n+2}}$ as defined in Example 2.5 and put $Y = \prod_{n<\omega} Y_n$. For each $n < \omega$, let π_n be the projection map onto Y_n.

Let $f \in Y$. For every $n < \omega$, $f(n)$ is not a $\mathscr{COZ}_{\omega_{n+1}}$-point of Y_n (by the previous example), so let C_n be a cozeroset of Y_n such that $f(n) \in \overline{C_n} \setminus G_{\omega_n}\text{-int}(\overline{C_n})$. For every $n < \omega$, let $\langle C_n^k : k < \omega\rangle$ be an increasing sequence of cozerosets of Y_n such that $C_n = \bigcup_{k<\omega} C_n^k$ and such that for every $k < \omega$, $\overline{C_n^k} \subset C_n$. For every $k < \omega$, define $D_k = \bigcap_{i\leq k} \pi_k^{-1}(C_i^k)$. Put $D = \bigcup_{k<\omega} D_k$. Then D is a cozeroset of Y and $f \in \overline{D}$.

Striving for a contradiction, assume that f is a $\mathscr{COZ}_{\omega_\omega}$-point and choose $n < \omega$ and a closed G_{ω_n}-set Z with $f \in Z \subset \overline{D}$. For each $k < \omega$, choose a closed G_{ω_n}-set Z_k of Y_k such that $f \in \prod_{k<\omega} Z_k \subset Z$. Z_n cannot be

contained in $\overline{C_n}$, so let $y \in Z_n \backslash \overline{C_n}$. Define $g \in Y$ by

$$g(k) = \begin{cases} f(k) & \text{if } k \neq n, \\ y & \text{if } k = n. \end{cases}$$

Then $g \in \prod_{k<\omega} Z_k$ and so $g \in \overline{D}$. But this is a contradiction as $\pi_n^{-1}(Y_n \backslash \overline{C_n}) \cap \bigcap_{i<n} \pi_i^{-1}(Y_i \backslash \overline{C_i^n})$ is a neighbourhood of g which is disjoint from $D = \bigcup_{k<\omega}(\bigcap_{i \leq k} \pi_k^{-1}(C_i^k))$. □

It might be possible to replace the condition $\kappa = \kappa^\omega$ in the Cozero Point Theorem by the weaker condition $\mathrm{cf}(\kappa) > \omega$, but we do not know.

Question 2.8. If X is a compact space with $w(X) = \kappa$ and $\mathrm{cf}(\kappa) > \omega$, then does X have a $\mathscr{COZ}_\kappa$-point? (Of course, this is true if one assumes GCH; this question is particularly interesting for $\kappa = \omega_1$.)

We do have a positive answer to this question provided X is a dyadic space.

Theorem 2.9 (Dyadic Cozero Point Theorem). *If X is a dyadic space and $w(X) = \kappa$ and $\mathrm{cf}(\kappa) > \omega$, then there exists a $\mathscr{COZ}_\kappa$-point in X.*

Proof. Let $\phi\colon 2^\kappa \to X$. For each $\alpha < \kappa$, let $\pi_\alpha\colon 2^\kappa \to 2^\alpha$ be the projection map. We say that a subset A of 2^κ *depends on* α if $A = \pi_\alpha^{-1}\pi_\alpha(A)$. We will define, by induction on α, where $\omega \leq \alpha < \kappa$, a sequence of closed $G_{|\alpha|}$-sets Z_α of X. Put $Z_\omega = X$. At stage α put $W = \bigcap_{\beta<\alpha} Z_\beta$. Then W is a closed $G_{|\alpha|}$-set. Put $Z_\alpha =$ some nonempty closed $G_{|\alpha|}$-set $Z \subset W$ such that if Z' is any nonempty closed $G_{|\alpha|}$-set with $Z' \subset Z$, then $\pi_\alpha(\phi^{-1}(Z')) = \pi_\alpha(\phi^{-1}(Z))$. Such a Z exists since 2^α has no strictly decreasing $|\alpha|^+$ sequence of closed sets.

At the end of this induction, by compactness, choose $x \in \bigcap_{\alpha<\kappa} Z_\alpha$. Then x is a $\mathscr{COZ}_\kappa$-point in X. To see this, assume that $B \in \mathscr{COZ}$ and that $x \in \overline{B}$. Then $\phi^{-1}(B)$ is a cozeroset in 2^κ and since $\mathrm{cf}(\kappa) > \omega$, we can choose an $\alpha < \kappa$ such that $\phi^{-1}(B)$ depends on α. From $x \in Z_\alpha \cap \overline{B}$, it follows that $\phi^{-1}(Z_\alpha) \cap \overline{\phi^{-1}(B)} \neq \varnothing$.

We claim that $Z_\alpha \subset \overline{B}$. If not, then choose $y \in Z_\alpha \backslash \overline{B}$. Choose a zeroset $A \ni y$ such that $A \cap \overline{B} = \varnothing$. Put $Z' = A \cap Z_\alpha$. Z' is a nonempty closed $G_{|\alpha|}$-set with $Z' \subset Z_\alpha$. So at stage α we have that $\pi_\alpha(\phi^{-1}(Z')) = \pi_\alpha(\phi^{-1}(Z_\alpha))$. $Z' \cap \overline{B} = \varnothing$ implies that $\phi^{-1}(Z') \cap \overline{\phi^{-1}(B)} = \varnothing$. Since $\phi^{-1}(B)$ depends on α, we get that $\overline{\phi^{-1}(B)}$ also depends on α. So, $\pi_\alpha(\phi^{-1}(Z')) \cap \pi_\alpha(\overline{\phi^{-1}(B)}) = \varnothing$. Hence, $\pi_\alpha(\phi^{-1}(Z_\alpha)) \cap \pi_\alpha(\overline{\phi^{-1}(B)}) = \varnothing$. Thus, $\phi^{-1}(Z_\alpha) \cap \overline{\phi^{-1}(B)} = \varnothing$—a contradiction. □

3. Applications to homogeneity

The first application is to resolve a question posed by the author in [Be2]. Let X be a zero-dimensional compact space, and let $\mathscr{CO}$ be the family of clopen subsets of X. A family of sets $\mathscr{A}$ is *centered* if for every finite $\mathscr{F} \subset \mathscr{A}$ we have that $\bigcap \mathscr{F} \neq \varnothing$. Define $\mathrm{Cen}(X) = \{f \in 2^{\mathscr{CO}}\colon \{B \in \mathscr{CO}\colon f(B) = 1\}$ is a centered family$\}$. $\mathrm{Cen}(X)$ is closed in $2^{\mathscr{CO}}$, hence it is compact. For each $B \in \mathscr{CO}$, put $B^* = \{f \in \mathrm{Cen}(X)\colon f(B) = 1\}$. Then B^* is a clopen subset of $\mathrm{Cen}(X)$. We consider $\mathrm{Cen}(\omega^*)$, where ω^* is the Stone-Cech remainder

$\beta\omega\backslash\omega$. In the above paper we investigated $\mathrm{Cen}(\omega^*)$, showed that it is in some sense the largest "Combinatorial" remainder of ω, and mentioned that every point has π-character c. We asked whether $\mathrm{Cen}(\omega^*)$ was homogeneous.

Proposition 3.1. $\mathrm{Cen}(\omega^*)$ *is not homogeneous.*

Proof. Let $\overline{0}$ be the constant function 0 in $\mathrm{Cen}(\omega^*)$. Let $\mathscr{B}$ be a disjoint family of clopen sets in ω^* of cardinality c. The disjoint clopen family $\{B^*: B \in \mathscr{B}\}$ witnesses the fact that $\overline{0}$ is a W_c-point; thus, by Proposition 2.3, $\overline{0}$ is not a $\mathscr{COZ}_c$-point. By the Cozero Point Theorem, since $w(\omega^*) = c = c^\omega$, $\mathrm{Cen}(\omega^*)$ must have a $\mathscr{COZ}_c$-point. Since having a $\mathscr{COZ}_c$-point is a topological invariant, $\mathrm{Cen}(\omega^*)$ is not homogeneous and our proposition is proved. □

It follows from the Point Theorem that ω^* has a $\mathscr{COZ}_c$-point. An interesting question which we have not been able to resolve without assuming extra set-theoretic assumptions is the following:

Question 3.2. Does ω^* have a point which is **NOT** a $\mathscr{COZ}_c$-point?

Now, let us restate the Point Theorems for a homogeneous compact space. A family $\mathscr{C} \subset \mathscr{P}(X)$ is called *invariant* if for all $C \in \mathscr{C}$ and for all autohomeomorphisms h of X we have that $h(C) \in \mathscr{C}$. In this case the existence of one point gives us that all points are $\mathscr{C}_\kappa$-points.

Theorem 3.3 (Homogeneous Theorem). *If X is a homogeneous compact space and $\mathscr{C}$ is an invariant family with $|\mathscr{C}| = \kappa$, then for each $C \in \mathscr{C}$, $\overline{C}$ is $G_{<\kappa}$-open in X.*

Theorem 3.4 (Cozero Homogeneous Theorem). *If X is a homogeneous compact space and $w(X) = \kappa = \kappa^\omega$ and $C \in \mathscr{COZ}$, then $\overline{C}$ is $G_{<\kappa}$-open in X.*

The Cozero Homogeneous Theorem cannot be improved by replacing $G_{<\kappa}$-open by a $G_{<\kappa}$-set.

Example 3.5. There exists a homogeneous compact space X such that

- $w(X) = c$;
- X has a cozeroset C such that $\overline{C}$ is not a $G_{<c}$-set.

Proof. Let Y be the Alexandroff Double of the Cantor set 2^ω. Then Y is a first countable, zero-dimensional, compact space with a dense set of isolated points. Put $X = Y^\omega$. By a result of Motorov [Mo], X is homogeneous. Let D be a countable, dense subset of 2^ω, and let E be the isolated points of Y which correspond to D. Then E is a cozeroset of Y and it is seen that $\overline{E}$ is not a $G_{<c}$-set in Y. Let π be the projection map of X onto the first factor space Y. It follows that if we put $C = \pi^{-1}(E)$, then C is a cozeroset of X and $\overline{C}$ is not a $G_{<c}$-set in X. □

However, in a dyadic space, the Cozero Homogeneous Theorem can be improved to

Theorem 3.6 (Dyadic Cozero Homogeneous Theorem). *If X is a homogeneous dyadic space and $w(X) = \kappa^+$ with $\kappa \geq \omega$ and $C \in \mathscr{COZ}$, then $\overline{C}$ is a G_κ-set.*

Proof. Applying homogeneity to the Dyadic Cozero Point Theorem we immediately get that if $C \in \mathscr{COZ}$, then $\overline{C}$ is G_κ-open in X. In dyadic spaces, closed G_κ-open sets are actually G_κ-sets, so our result follows. A justification of this

just-quoted dyadic fact goes as follows: Theorem 6 of Engelking [En1, p. 293] implies that in a Cantor cube, a closed G_κ-open set is actually a G_κ-set. The general dyadic case follows from an easy argument showing that this property is preserved under closed continuous images. □

Efimov [Ef] asked whether there existed a zero-dimensional, homogeneous dyadic space not homeomorphic to 2^κ. This was answered by Pashenkov [Pa], who showed that they do exist. However, the smallest weight example that he produced was of weight 2^{ω_1}. Using the Cozero Homogeneous Theorem plus the additional assumption of the Continuum Hypothesis, we showed that there could be no such example of weight ω_1. Subsequently, L. Shapiro [Sh] removed the CH assumption completely from this result. After being informed of this, we proved the Dyadic Cozero Homogeneous Theorem. The reader is encouraged to read [Sh] for an inverse limit approach to this problem.

Corollary 3.7. *The Cantor cube 2^{ω_1} is topologically the only zero-dimensional, homogeneous dyadic space of weight ω_1.*

Proof. Let X have all the properties in the corollary. Since $\chi(X) = w(X)$ in dyadic spaces and X is homogeneous, we get that X has uniform character ω_1. Since, in a dyadic space, a regular closed set is the closure of some cozeroset, by applying the preceding theorem, we get that every regular closed set is a G_ω-set.

To complete the proof, we now use the following result of Schepin [Sc]: A zero-dimensional dyadic space of uniform character ω_1 in which every regular closed set is a G_ω-set is homeomorphic to 2^{ω_1}. As this result does not explicitly appear in this reference, to help our readers, we explain it using notions and theorems from that paper. Theorem 18 on p. 10 implies that X is open-generated. Since $w(X) = \omega_1$, Theorem 3 on p. 2 implies that X is AE(0). Finally, Theorem 1 on page 1 implies that X is a Cantor cube. □

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