BRANCHED SURFACES AND THURSTON'S NORM ON HOMOLOGY

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ABSTRACT. Given a closed, irreducible, orientable 3-manifold M , let x denote the Thurston norm on $H_2(M; R)$. Suppose g , h , and f are three homology classes of $H_2(M; Z)$ carried by a single face of the x -unit sphere in $H_2(M; R)$. In this paper it is shown that there exists a taut, oriented branched surface carrying representatives of g and h and a semi-taut oriented branched surface carrying representatives of all three homology classes.

Throughout, M will be a closed, irreducible, orientable 3-manifold. In [T] Thurston introduced a semi-norm x on $H_2(M; R)$. For $g \in H_2(M; Z)$, $x(g)$ is defined to be the infimum, taken over all embedded surfaces G representing g , of $\chi_-(G) = \sum_i |\chi(G_i)|$ where the sum is over all components G_i of G with $\chi(G_i) \leq 0$. Then x is extended to all of $H_2(M; R)$ using homogeneity and continuity. The x -unit ball $\mathcal{B}_x$ is a finite polyhedron which is compact when M is also atoroidal. Oertel, in [O], showed that there exists a finite collection of taut, oriented, homology branched surfaces (which are RIB's) carrying χ_- -minimizing representatives of every homology class of $H_2(M; Z)$. All homology classes represented by surfaces which are not tori and are carried by a given semi-taut oriented branched surface are carried by a single face of the x -unit sphere $\partial\mathcal{B}_x$. The question arises as to how the faces of $\partial\mathcal{B}_x$ relate to taut (or semi-taut) oriented branched surfaces and has been considered in [M], [O], and [S].

An incompressible surface S in a closed irreducible 3-manifold M is an embedded surface without any 2-sphere components satisfying the condition that if a simple closed curve in S bounds a disc in M then it bounds a disc in S . An oriented surface S is said to be χ_- -minimizing if it has no 2-sphere components, $[S] \neq 0$, and $x([S]) = -\chi(S)$ where x is Thurston's norm on $H_2(M, \partial M)$, $[S]$ is the homology class represented by S , and $\chi(S)$ is the Euler characteristic. A χ_- -minimizing surface may contain compressible or homologically trivial torus components. In the literature, a norm-minimizing surface usually refers to an incompressible, χ_- -minimizing surface.

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Let σ be a face of the x -unit sphere in $H_2(M; R)$. Oertel [O] proved that given two classes g, h of $H_2(M; Z) \subset H_2(M; R)$ carried by σ , there exists a semi-taut oriented branched surface carrying χ_- -minimizing representatives of g and h .

An oriented branched surface is *semi-taut* if it carries only χ_- -minimizing surfaces (such branched surfaces may carry compressible tori or homologically trivial tori). We define a *taut oriented branched surface* B as an oriented branched surface which carries only incompressible, χ_- -minimizing surfaces (the possibility that it carries homologically trivial tori or homologically trivial disjoint unions of tori is not excluded). We show that given two classes g, h of $H_2(M; Z)$ carried by σ , there exists a taut, oriented branched surface Σ carrying representatives of both g and h . It is unclear whether or not this result can be strengthened to obtain a taut branched surface that does not carry any homologically trivial disjoint unions of tori (or even homologically trivial tori). We also show that given three classes g, h, k of $H_2(M; Z)$ carried by σ , there exists a semi-taut oriented branched surface Σ carrying representatives of g, h , and k . An example of Sterba-Boatwright [S] demonstrates that, in general, one cannot always find a taut, oriented branched surface carrying representatives of all three classes g, h , and f , or even carrying representatives of their projective classes in $\partial\mathcal{B}_x$. This leaves open the question posed by Oertel as to whether or not one can always find a semi-taut branched surface carrying representatives of all integral homology classes in the cone of σ .

The use of the term *taut branched surface* in the literature is not consistent. Our definition is the same as in [S] and seems the most suitable. Our definition of a semi-taut branched surface coincides with that of a taut branched surface in [O] and, by Corollary 1 and Example 1, is equivalent to that of a taut branched surface in [M].

For an oriented branched surface B there is a consistent orientation on the I -fibers of its fibered regular neighborhood $N(B)$. We let $\pi : N(B) \rightarrow B$ denote the projection map which collapses fibers to points. A surface F is *carried by* B if it can be embedded in $N(B)$ transverse to the fibers.

Given an orientation on M , a unique orientation is induced on F by the oriented fibers of $N(B)$. The closures in B of the complements of the branch loci are called the *sectors* of B . If we let $\vec{w} = w(F)$ assign to each sector of B the number of points in which F intersects a fiber over the sector, then F can be uniquely associated, up to isotopy, with $\vec{w}$ and will be denoted by $B(\vec{w})$. Given any such assignment $\vec{w}$ of a nonnegative integer w_i to each sector of B satisfying the branch equations, there exists a surface $B(\vec{w})$ carried by B . Basic definitions and results related to branched surfaces can be found in [O].

Lemma 1. *Let $F = B(\vec{w})$ be an oriented surface carried by an oriented branched surface B . If there exists an oriented submanifold W of $M - F$ such that $F = \partial W$, as oriented surfaces, then the invariant measure $\vec{w}$ assigns a weight of either 0 or 1 to each sector.*

Proof. Suppose that there exists a fiber I of $N(B)$ such that $F \cap I$ contains two or more points. Choose a and b to be adjacent points of $F \cap I$ in I . The oriented fiber I contains an arc with endpoints $\{a, b\}$ lying entirely in either W or $\overline{M - W}$. It follows that the orientation of $F = \partial W$ at either a or b is inconsistent, a contradiction. $\square$

If G and H are oriented surfaces intersecting transversely, the cut-and-paste operation along an intersection curve which preserves the given orientations is called an *orientation-preserving exchange*. The *orientation sum* $G + H$ is obtained by performing an orientation-preserving exchange along each intersection curve. For each component α of $G \cap H$ we can associate an annulus A in a small regular neighborhood of α such that α is a core of A and $A \cap (G + H) = \partial A$. We refer to the two boundary curves of A as *trace curves* corresponding to the orientation-preserving exchange along α . The closures of the components of the complement of the trace curves in $G + H$ are called *patches*. Observe that if G and H are carried by an oriented branched surface B then so is $G + H$ and $\vec{w}(G + H) = \vec{w}(G) + \vec{w}(H)$.

Lemma 2. *Let S be a closed surface bounding an oriented submanifold V in M and assume S is oriented as the boundary of V . Let W be a regular neighborhood of V in M . Suppose X is an oriented surface properly embedded in W which intersects S transversely and has the property that each component of $X \cap V$ separates V . Then there exists at least one closed component of $X + S$ contained in W .*

Proof. It is sufficient to consider only the case when V is connected. We use induction on the number of components of $X \cap V$. First suppose that $X \cap V$ has only one component C . Let Y_1 and Y_2 denote the closures of the components of $V - C$ with orientations chosen such that the orientation of ∂Y_j agrees with the orientation of $C \subset \partial Y_j$. One of the Y_j , say Y_1 , has an orientation agreeing with that of V . It follows that the closed surface ∂Y_1 is a component of $X + S$ and is contained in W .

Now assume that $X \cap V$ has n components, denoted by $C_1, \dots, C_n$, and that the lemma holds whenever there are fewer than n components. Call a component C_i innermost in V if there is a component Y of $V - C_i$ that does not contain any C_j .

Since each C_i separates V , there exists an innermost component of $X \cap V$, say C_1 . Let Y and V' denote the closures of the two components of $V - C_1$, where Y is the one which does not contain any C_j for $j = 2, \dots, n$. As before, we orient Y and V' such that the induced orientation on their boundaries agrees with the orientation of C_1 . If the orientation of Y agrees with the orientation of V then ∂Y is a closed component of $X + S$ and the conclusion follows.

If the orientation of Y does not agree with that of V then the orientation of V' agrees with that of V and we let $S' = \partial V'$. Choose W' to be a regular neighborhood of V' contained in W . We may assume that there is only one component $\hat{C}_1$ of $X \cap W'$ containing C_1 and we let $X' = X - \hat{C}_1$. Now $C_2, \dots, C_n$ are the components of $X' \cap V'$ and, by our induction hypothesis, $X' + S'$ has a closed component Z contained in W' . Observe that we can view $C_1 + S$ as the disjoint union of S' and a bounded surface T' . Hence $X + S = (X' + S') \cup T'$ and clearly Z is a component of $X + S$. $\square$

Corollary 1. *If the oriented branched surface B carries an incompressible, χ_- -minimizing surface K for which $w(\vec{K})$ is positive on each sector, then B is a semi-taut branched surface.*

Proof. The first thing to note is that B cannot carry a 2-sphere. If S is a 2-sphere carried by B , then, by Lemma 1, we can write $K = S + X$, where X is also carried by B and $\bar{w}(X) = \bar{w}(K) - \bar{w}(S)$. Let W be a small regular neighborhood of the 3-cell in M bounded by S . It follows from Lemma 2 that K has a component contained in W . But this is impossible since K is incompressible.

To see that B is semi-taut we consider a surface F_1 carried by B . For some positive integer n we have $n\bar{w}(K) = \bar{w}(F_1) + \bar{w}(F_2)$, where $F_1 + F_2$ consists of n copies of K and neither F_1 nor F_2 have any 2-sphere components. Suppose there exists a surface F'_1 homologous to F_1 with $\chi_-(F'_1) < \chi_-(F_1)$. We may assume that F'_1 has no 2-sphere components. This leads us to the following contradiction: $x(n[K]) = n\chi_-(K) \leq \chi_-(F'_1 + F_2) \leq \chi_-(F'_1) + \chi_-(F_2) < \chi_-(F_1) + \chi_-(F_2) = \chi_-(F_1 + F_2) = x(n[K])$. Therefore, F_1 is χ_- -minimizing. $\square$

Given oriented surfaces G and H intersecting transversely, we can form an oriented branched surface from $G \cup H$ by flattening out $G \cup H$ along each intersection curve to form an annulus band of contact. This flattening is done in such a way that the orientations of G and H agree along each annulus of contact. The resulting branched surface, denoted by $B(G, H)$, has only simple branched loci and carries G , H , and $G + H$.

Some simple examples can be constructed using $M = F \times S^1$ where F is a closed orientable surface of genus two. Let $G = F \times \{0\}$ and T be the torus boundary of $D \times S^1$ where D is any small disk in G . We set $T_i = \alpha_i \times S^1$ where $\alpha_1, \alpha_2, \alpha_3$, and α_4 are simple closed curves generating $H_1(F)$. Finally, take $S = c \times S^1$ where c is an essential simple closed curve in F separating F . Fix orientations on M , G , S , T , and the T_i .

Example 1. Let B denote the oriented branched surface obtained by flattening $G \cup T$ along the single intersection curve $G \cap T$. Observe that $G + T$ is isotopic to the incompressible, χ_- -minimizing surface G and is carried by B with $\bar{w}(G + T)$ positive on both sectors. However, B also carries the compressible torus T , referred to as a Reeb component in [O]. Thus, one cannot conclude (as implied in [M]) that if an oriented branched surface B carries an incompressible, χ_- -minimizing surface K for which $\bar{w}(K)$ is positive on each sector then B carries only incompressible surfaces.

Example 2. $H_2(M; R) = R^5$ where we may assume that $[G]$ generates $R \times 0$ and the homology classes of the four tori T_i form a basis for $0 \times R^4$. Observe that there exist two faces, both noncompact, for the x -unit sphere $\partial B_x = S^0 \times R^4$. Homology classes of the form $[G + T_1]$ are carried by ∂B_x while those of the form $[T_1 + T_2]$ are not. The line through $[G]/x([G])$ and $[G + T_1]/x([G + T_1])$ never meets the boundary (which is empty) of the face of ∂B_x containing it.

Theorem 1. Let g and h be homology classes of $H_2(M; \mathbb{Z})$ carried by a face of $\partial \mathcal{B}_x$. Then there exists a taut, oriented branched surface carrying g and h .

Proof. Let G and H be oriented incompressible, χ_- -minimizing surfaces which are representatives of g and h , respectively. Assume that G and H intersect transversely and that $G \cap H$ has the fewest possible number of components relative to such surfaces representing g and h . Since g and h are

carried by a face of the x -unit sphere, we have

$$x([G] + [H]) = x([G]) + x([H]).$$

Observe that G and H can have no contractible intersection curves because of our assumption on $G \cap H$.

Let B denote the oriented branched surface $B(G, H)$ obtained from G and H by flattening along the intersections. Since the branched surface B has only simple branched loci, corresponding to the components of $G \cap H$, there can be no disk sectors and hence B cannot carry a 2-sphere. From this it follows that $x([G + H]) = x([G]) + x([H]) = \chi_-(G) + \chi_-(H) = -\chi(G) - \chi(H) = -\chi(G + H) = \chi_-(G + H)$. Thus $G + H$ is a χ_- -minimizing representative of $g + h$.

Next we show that $G + H$ is incompressible. Since $x([G + H]) = -\chi(G + H)$ and $G + H$ has no 2-sphere component, we have $G + H = F \cup T$, where F is an incompressible, χ_- -minimizing surface and T is a pairwise disjoint collection of compressible tori disjoint from F . Let $A_1, B_1, \dots, A_n, B_n$ denote the consecutive annuli patches in one component X of T , where notation is chosen such that $A_i \subset G$ and $B_i \subset H$. Since the orientation sum has X as a component, the orientations on $\cup A_i \subset G$ and $\cup B_i \subset H$ are consistent along $\cup \partial A_i = \cup \partial B_i$ and agree with the orientation of X . Thus $G' = (G - \cup A_i) \cup (\cup B_i)$ and $H' = (H - \cup B_i) \cup (\cup A_i)$ are surfaces homologous to G and H , respectively. Moreover, since $\chi(G') = \chi(G)$, $\chi(H') = \chi(H)$ and neither G' nor H' have 2-sphere components, it follows that G' and H' are each disjoint unions of incompressible, χ_- -minimizing surfaces G'', H'' and compressible tori, where G'' is homologous to G and H'' is homologous to H . Now $G'' \cap H''$ has fewer components than $G \cap H$, contradicting our choice of G and H . We conclude that $G + H = F$ and hence is incompressible and χ_- -minimizing. It follows from Corollary 1 that B is a semi-taut oriented branched surface.

To show that the branched surface B is taut, it is sufficient to show that it cannot carry a compressible torus. Suppose that there exists such a torus T carried by B . Let D be an essential compression disc for T and choose $D \times I$ in a small regular neighborhood of D such that $D = D \times 1/2$ and $(D \times I) \cap T = \partial D \times I$. The 2-sphere $S = (T - \partial D \times I) \cup (D \times \partial I)$ bounds a 3-cell X . We have two cases:

- (i) $D \subset \overline{M - X}$, in which case T bounds a solid torus $W = X \cup (D \times I)$;
- (ii) $D \subset X$, in which case T bounds a knot complement $W = \overline{X - D \times I}$.

It follows from Lemma 1 that $w_i(T) \leq 1$ for each i . Thus there exists a surface F carried by B such that $G + H = F + T$. We may assume that T is chosen, up to isotopy in $N(B)$, such that F and T intersect transversely and the number of components of $F \cap T$ is as small as possible. Note that $\bar{w}(G + H) = \bar{w}(F) + \bar{w}(T)$ assigns 2 to the flattened annulus of contact sectors of B and 1 to all other sectors. It follows that $F \cap T$ is contained in the subbundle $N(A)$ of the fibered neighborhood $N(B)$ over the annulus of contact sectors.

Suppose a component α of $F \cap T$ is contractible. If α were contractible in $N(A)$ then it could be removed by an isotopy of T in $N(B)$. If α were not contractible in $N(A)$ then this would imply that B has a contractible branch

locus. Thus no component of $F \cap T$ is contractible. Another fact we use is that each component of $F \cap (M - T)$ is a subsurface of the oriented incompressible surface $G + H$.

Case (i): W is a solid torus. In this case $F \cap W$ is a pairwise disjoint collection of essential annuli in W and, in particular, each component of $F \cap W$ separates W . We see from Lemma 2 that some component of $F + T = G + H$ is contained in a regular neighborhood of W and hence is compressible. But this is impossible since $G + H$ is incompressible.

Case (ii): W is the knot complement $\overline{X - D \times I}$. Without loss of generality, we may assume that the compression disk D has been chosen such that the following properties hold:

(a) Each component of $(D \times I) \cap F$ has the form $\lambda \times I$, where λ is an arc properly embedded in D .

(b) For each component α of $F \cap T$, the intersection signs between α and ∂D are the same at each of the points $\alpha \cap \partial D$.

(c) If λ is a component of $D \cap F$ joining components α and β of $F \cap T$, then $\alpha \neq \beta$.

If there exist components not satisfying condition (b), then one can find an innermost disk e in T with ∂e the union of two arcs, one in ∂D and the other in a component of $F \cap T$. By isotopying D across e we can reduce the number of components of $F \cap D$. Condition (c) is a consequence of (b). Orient each component of $F \cap T$ as a boundary curve of $F \cap (M - \overset{\circ}{W})$. If there exists a component λ of $D \cap F$ joining α and β then the subdisk $\lambda \times I$ of $F \cap (M - \overset{\circ}{W})$ determines orientations of α and β which would violate (b) if $\alpha = \beta$.

To conclude, observe that each component E of $F \cap X$ is a disk-with-holes and each boundary component of E must intersect $D \times \partial I$. Components of $F \cap W$ are obtained from these disks-with-holes E by deleting neighborhoods $\lambda \times I \subset D \times I$ of arcs $x \times I$ properly embedded in E . Hence $F \cap W = F \cap X - \cup(\overset{\circ}{\lambda} \times I)$, where λ ranges over the components of $D \cap F$. Choose a disk-with-holes E and the family $\Lambda = \{\lambda\}$ of properly embedded arcs whose neighborhoods are deleted in passing to $E \cap W$. From properties (b) and (c) we see that every boundary curve of E meets some arc in Λ and none of the arcs λ have both endpoints in a single boundary component of $E - \cup(\lambda \times \overset{\circ}{I})$. A simple induction on the number of boundary curves of E shows that that one component of $E - \cup(\lambda \times \overset{\circ}{I})$ is a disk. If every λ meets only one component of E , the conclusion is clear. If some λ meets two distinct boundary curves of some component E then replace E by $E - (\lambda \times \overset{\circ}{I})$ and apply the induction hypothesis. From this it follows that some component of $F \cap W$ is a disk and hence W is a solid torus. As this case has already been treated in (i), we can conclude that no compressible tori are carried by B . $\square$

Remark. Theorem 1 allows us to repair a gap in the proof of Theorem 9 in [O]. In that proof, Oertel finds a surface F homologous to a fiber and such that F is carried by a branched surface B obtained by using his Proposition 8. For the proof to work, it is necessary that F be incompressible. However, the proof of Proposition 8 allows F to have compressible torus components. To see this,

consider a situation in which there is a solid torus $U = I \times I \times S^1$ such that $G \cap H = \partial I \times \partial I \times S^1$ with $G \cap U = I \times \partial I \times S^1$ and $H \cap U = \partial I \times I \times S^1$. If the orientations are right, pinching $G \cup H$ together to form a branched surface, as in the proof of Proposition 8, produces a branched surface carrying G and H as well as the compressible torus ∂U . The branched surface necessary for the proof of Theorem 9 to work is provided by our Theorem 1.

Theorem 2. *Let g, h , and k be homology classes of $H_2(M; \mathbb{Z})$ carried by a face of $\partial \mathcal{B}_x$. Then there exists a semi-taut oriented branched surface carrying g, h , and k .*

Proof. We first choose incompressible, χ_- -minimizing representatives G and H of g and h , respectively. We may assume that G intersects H transversely and the number of components of $G \cap H$ is as small as possible among all such incompressible, χ_- -minimizing representatives. By Theorem 1, the oriented branched surface obtained from $G \cup H$ by flattening the intersection curves is taut. Let $K = G + H$. Choose an incompressible, χ_- -minimizing representative F of f such that F intersects $G \cup H$ and K transversely.

We may choose F such that $F \cap K$ does not contain any contractible components. Suppose that $F \cap K$ contains a contractible component. Since both K and F are incompressible, we can find a component α which bounds an innermost disk D in K and a disk E in F . Take $F_1 = (F - E) \cup D$ and move F_1 off K at α , keeping it transverse to G and H . Since M is irreducible, F_1 is isotopic to F . Repetition of this process produces a surface, which we continue to denote by F , that is an incompressible, χ_- -minimizing representative of f and such that $F \cap K$ does not have any contractible components. Thus there are no contractible trace curves for the sum $K + F$ and it follows that $K + F$ does not have any 2-sphere components.

Now let B denote the branched surface obtained from $G \cup H \cup F$ by flattening their intersections. This is illustrated locally in Figure 1 (see p. 642). We first show that B cannot carry a 2-sphere. Suppose there exists a 2-sphere S carried by B . By Lemma 1, there is a surface X carried by B such that $G + H + F = K + F = S + X$. Let W be a regular neighborhood of the 3-cell in M bounded by S . By Lemma 2, $S + X$ has a closed component Z contained in W . Since W is a ball, either Z is a component of K or F , or contains contractible trace curves of $K + F$. Since both of these cases are impossible, it follows that B does not carry any 2-spheres. Using this together with the fact that g, h , and f all lie in the same face of the $\partial \mathcal{B}_x$, we have

$$\begin{aligned} x(g + h + f) &= x(g) + x(h) + x(f) \\ &= -\chi(G) - \chi(H) - \chi(F) = -\chi(G + H + F). \end{aligned}$$

Therefore $G + H + F$ is a χ_- -minimizing representative of $g + h + f$.

To see that B is semi-taut we consider a surface X carried by B . There is a positive integer n and a surface Y carried by B such that

$$n(G + H + F) = X + Y.$$

Observe that $X + Y$ is a χ_- -minimizing representative of $n(g + h + f)$. Since B does not carry 2-spheres, there can be no 2-sphere components of X, Y , or $X + Y$. Hence $\chi_-(X + Y) = \chi_-(X) + \chi_-(Y)$. If X is not χ_- -minimizing

FIGURE 1. The oriented branched surface B near a triple point

then there is a surface X' homologous to X such that $\chi_-(X') < \chi_-(X)$. We may assume that X' intersects Y transversely and $X' \cap Y$ does not contain any contractible components. Thus $X' + Y$ has no 2-sphere components and hence $\chi_-(X' + Y) = \chi_-(X') + \chi_-(Y)$. Then we have

$$x([X + Y]) \leq \chi_-(X' + Y) = \chi_-(X') + \chi_-(Y) < \chi_-(X) + \chi_-(Y) = \chi_-(X + Y)$$

which contradicts that $x([X + Y]) = \chi_-(X + Y)$. Therefore, X is χ_- -minimizing and thus B is a semi-taut oriented branched surface. $\square$

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