EVERY 3-MANIFOLD WITH BOUNDARY EMBEDS IN $\text{Triod} \times \text{Triod} \times I$

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ABSTRACT. Let M be a compact, connected 3-manifold with nonempty boundary. Then M embeds in $T \times T \times I$, where T is a triod and $I = [0, 1]$.

1. INTRODUCTION

Gillman and Rolfsen [2] proved that every compact, orientable 3-manifold with boundary embeds in $c(p) \times c(q) \times c(r)$, where $c(n)$ denotes the cone on n points. In this paper, we improve their argument to the following:

Theorem. *Let M be a compact, connected 3-manifold with nonempty boundary. Then M embeds in $\text{Triod} \times \text{Triod} \times I$.*

We use the term fake 3-ball for a compact, contractible 3-manifold which is not homeomorphic to the 3-ball I^3 ; and let $\cong$ and $\hookrightarrow$ denote homeomorphism and embedding, respectively. According to the theorem, the 3-dimensional Poincaré conjecture is equivalent to (also see [2]):

Conjecture. *There is no fake 3-ball in $T \times T \times I$.*

Gillman [1] proved that if a compact 3-dimensional M embeds in $c(p) \times I \times I$ and has trivial rational homology, then $M \cong I^3$. This fact, indicates that the above conjecture is true for 3-manifolds in $T \times I \times I$. On the other hand, our theorem answers Gillman's question in [1] in the negative.

2. A LEMMA

Let $K = T \times I$, PQ be the binding $v \times I$ for the vertex v of order 3 of T and α, β, γ be the three pages (i.e., rectangles) of $T \times I$. We will use $A_i A_j$ to denote the line segment whose endpoints are A_i, A_j .

Definition. Suppose disjoint annuli $G_1, G_2, \dots, G_n$, lie in K . If $(\bigcup_{i=1}^n G_i) \cap PQ$ is some line segment and each of the line segments has a regular neighborhood which consists of two discs such that each of the discs exactly lie in a page, then we say that $G_1, G_2, \dots, G_n$ are transversal in K and call the discs of $(\bigcup_{i=1}^n G_i) \cap \alpha, (\bigcup_{i=1}^n G_i) \cap \beta, (\bigcup_{i=1}^n G_i) \cap \gamma$ band discs.

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FIGURE 1

Definition. Let annuli $G_1, G_2, \dots, G_n$ be transversal in K . Suppose that K lies in R^3 and α is above $\beta \cup \gamma$. Then we say that K lies in R^3 standardly and call $\bigcup_{i=1}^n \text{Bd } G_i$ a band link in $K \subset R^3$.

Definition. Let annuli $G_1, G_2, \dots, G_n$ be transversal in K and band disc $X \subset \alpha$. Suppose $\bigcup_{i=1}^n G_i$ does not intersect the shade region of Figure 1. Then we move X to X' as in Figure 1 to get new annuli. We call such a move an elementary move. Compositions of elementary moves are called I-moves.

Clearly, suppose that annuli $\bigcup_{i=1}^n G_i \subset K$ I-moves to $\bigcup_{i=1}^n G'_i$; then there is an embedding $H: (\bigcup_{i=1}^n G_i) \times I \rightarrow K \times I$ such that $H((\bigcup_{i=1}^n G_i) \times 0) = \bigcup_{i=1}^n G_i \subset K \times 0$ and $H((\bigcup_{i=1}^n G_i) \times 1) = \bigcup_{i=1}^n G'_i \subset K \times 1$.

Lemma. Let annuli $G_1, G_2, \dots, G_n$ be transversal in $K \times A_1$, where A_1 is a point. Then there are n disjoint 3-balls $B_1, B_2, \dots, B_n$ in $K \times (A_1 A_2 \cup A_2 A_3 \cup A_4 A_2)$ such that $G_i \subset \text{Bd } B_i$ ($i = 1, 2, \dots, n$).

Proof. Suppose that if we let $K \times A_1$ lie in R^3 standardly, then we only need change the crossing relations of m pairs of band discs $(X_1, X'_1), \dots, (X_m, X'_m)$ to get new annuli whose central curve is a trivial link in R^3 .

For band discs X_1, X'_1 , we have an I-move, I_1 , a pair of new band discs (Y_1, Y'_1) , and annuli $G'_i = I_1(G_i)$ ($i = 1, 2, \dots, n$) as in Figure 2. Note that we can replace changing the crossing relation between X_1 and X'_1 with changing the crossing relation between Y_1 and Y'_1 . For $I_1(X_2), I_1(X'_2)$, using a similar method, we get $I_2, (Y_2, Y'_2)$, and $G''_1, \dots, G''_n, \dots$. For $I^{m-1} I^{m-2} \dots I_1(X_m)$ and $I^{m-1} I^{m-2} \dots I_1(X'_m)$, we get $I_m, (Y_m, Y'_m)$, and $G^{(m)}_1, \dots, G^{(m)}_n$.

Let $I' = I^m I^{m-1} \dots I^2 I^1$. Then $G^{(m)}_i = I'(G_i)$ ($i = 1, 2, \dots, n$). We still use Y_i, Y'_i to denote $I_m I_{m-1} \dots I_i(Y_i), I_m I_{m-1} \dots I_i(Y'_i)$, respectively ($i = 1, 2, \dots, m$).

Suppose that $K \times A_1$ lies in R^3 standardly. If we change the crossing relations of the pairs $(Y_1, Y'_1), \dots, (Y_m, Y'_m)$ in R^3 to get new annuli $H_1, H_2, \dots, H_n$, then the central curve of new annuli is a trivial link in R^3 . Suppose that H_i has k_i twists in R^3 ($i = 1, 2, \dots, n$). Then we I-move $G^{(m)}_1, \dots, G^{(m)}_n$ to get annuli $L_1, L_2, \dots, L_n$ as Figure 3. We denote this I-move as I'' and denote the shade band discs of Figure 3 as $Y_{m+1}, Y_{m+2}, \dots, Y_{m'}$ ($m' = m + k_1 + k_2 + \dots + k_n$). Note that in Figure 3, if we change the band discs $Y_1, Y_2, \dots, Y_{m'}$ of $L_1, L_2, \dots, L_n$ to under $\beta \cup \gamma$ in R^3 , then we obtain annuli whose boundary curve is a trivial link in R^3 .

FIGURE 2

FIGURE 3

Let I -move $I = I''I'$. Then there is an embedding $H: (\bigcup_{i=1}^n G_i) \times A_1 A_2 \hookrightarrow K \times A_1 A_2$ such that

$$H \left(\left(\bigcup_{i=1}^n G_i \right) \times A_1 \right) = \left(\bigcup_{i=1}^n G_i \right) \subset K \times A_1$$

and

$$H \left(\left(\bigcup_{i=1}^n G_i \right) \times A_2 \right) = \left(\bigcup_{i=1}^n L_i \right) \subset K \times A_2$$

FIGURE 4

Consider $\alpha \times (A_2A_3 \cup A_4A_2) \cong I^3$. For $Y_1, Y_2, \dots, Y_{m'}$, we give band discs $Z_1, Z_2, \dots, Z_{m'}$ in $PQ \times A_4A_2$; for the other band discs of $(\bigcup_{i=1}^n L_i) \cap (\alpha \times A_2)$, we give band discs $Z_{m'+1}, Z_{m'+2}, \dots, Z_{m''} \subset PQ \times A_2A_3$. Then we get m'' small annuli in $\alpha \times (A_2A_3 \cup A_4A_2)$. Note that there are m'' disjoint 3-balls $B_1, B_2, \dots, B_{m''}$ in $\alpha \times (A_2A_3 \cup A_4A_2)$ such that the small annuli lie in the boundary surfaces of the 3-balls, respectively. (See Figure 4.)

Let $L' = (\bigcup_{i=1}^{m''} Z_i) \cup ((\bigcup_{i=1}^n L_i) \cap (\beta \cup \gamma))$. Then L' is n annuli $L'_1, L'_2, \dots, L'_n$ and whose boundary curve is a trivial link in $(\beta \cup \gamma) \times (A_2A_3 \cup A_4A_2) \cong I^3$. Therefore, there are n disjoint 3-balls $B'_1, B'_2, \dots, B'_n$ in $(\beta \cup \gamma) \times (A_2A_3 \cup A_4A_2)$ such that these annuli lie in the boundary surfaces of the 3-balls, respectively.

$H((\bigcup_{i=1}^n G_i) \times A_1A_2) \cup (\bigcup_{i=1}^{m''} B_i) \cup (\bigcup_{i=1}^n B'_i)$ are n disjoint 3-balls in $K \times (A_1A_2 \cup A_2A_3 \cup A_4A_2) \cong T \times T \times I$ and $G_1, G_2, \dots, G_n$ lie in the boundary surfaces of these 3-balls.

3. A PROOF OF THE THEOREM

We only need to prove the theorem in the case that $M \cong N - \text{Int } B$ for a closed 3-manifold N and a 3-ball $B \subset N$. If we let M be any compact 3-manifold with boundary and N be the closed 3-manifold given by attaching 3-dimensional handles to $\text{Bd } M$, then, by [3, Theorem 1.5], $M \subset N - \text{Int } C \cong N - \text{Int } B$, where C is a 3-ball in $N - \text{Int } M$ and B is any 3-ball in N .

Suppose that (V_1, V_2) is a Heegaard splitting of N . Let $(D_1, D_2, \dots, D_n)$ be a collection of pairwise disjoint property embedded 2-discs in V_1 which cut V_1 into a 3-ball, and let annuli $G_1, G_2, \dots, G_n$ be disjoint regular neighborhoods of the boundary curves $J_1, J_2, \dots, J_n$ of $D_1, D_2, \dots, D_n$ in $\text{Bd } V_1 = \text{Bd } V_2$, respectively.

We choose a proper 2-disc D in $\text{Bd } V_2$ such that D does not intersect the annuli. Then we have (see Figure 5) $V_2 \cong ((\text{Bd } V_2 - \text{Int } D) \times I) \cup (\bigcup_{i=1}^n E_i)$, where $E_1, E_2, \dots, E_n$ are 3-balls in V_2 and $(\bigcup_{i=1}^n \text{Bd } B_i) \cap ((\text{Bd } V_2 - \text{Int } D) \times 0)$ are annuli $F_1, F_2, \dots, F_n$.

FIGURE 5

Clearly, $(\text{Bd}V_2 - \text{Int}D) \times 0$ embeds in $K \times C_1$ for a point C_1 . Then we have $(\text{Bd}V_2 - \text{Int}D) \times I \hookrightarrow K \times C_1C_2$ such that annuli $G_1, G_2, \dots, G_n$ are transversal in $K \times C_2$ and annuli $F_1, F_2, \dots, F_n$ are transversal in $K \times C_1$. Therefore, by the lemma, $\bigcup_{i=1}^n E_i \hookrightarrow (K \times C_1A_1 \cup A_1A_2 \cup A_3A_1)$. Then, $M \hookrightarrow K \times C_5C_3 \cup C_3C_4 \cup C_2C_3 \cup C_1C_2 \cup C_1A_1 \cup A_1A_2 \cup A_3A_1) \cong T \times H \times I$. Note that $H \times I \hookrightarrow T \times I$. (This can be seen by drawing a picture.) Therefore, $M \hookrightarrow T \times T \times I$.

Remark. Mr. Dale Rolfsen told me the fact that $H \times I \hookrightarrow T \times I$.

Considering the proof of the theorem, we get the following corollary easily.

Corollary. Suppose that V_2 lies in R^3 trivially. Then $\bigcup_{i=1}^n \text{Bd}G_i$ and $\bigcup_{i=1}^n J_i$ are links in R^3 . If the band link $\bigcup_{i=1}^n \text{Bd}G_i$ is a trivial link, then $M \hookrightarrow T \times I \times I$; if $\bigcup_{i=1}^n J_i$ is a trivial link, then M has a standard spine which embeds in $T \times I \times I$.

In particular, by [1], the band link of any Heegaard splitting of a punctured lens space is not trivial in R^3 . On the other hand, $L(p, 1) - B$ has standard spine in $T \times I \times I$, where B is a 3-ball in $L(p, 1)$.

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