ON THE SOBOLEV CLASS OF A COMPOSITE FUNCTION

DENNIS D. COX AND FINBARR O'SULLIVAN

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ABSTRACT. It is shown that the Sobolev class of a function of the form $H(x, \theta(x))$ is the same as the Sobolev class of θ , for sufficiently smooth H . This result has applications in a perturbation analysis of a nonlinear system of differential equations considered elsewhere.

1. INTRODUCTION

Here we give a result concerning the Sobolev class of a composite function. Specifically, we consider a function $f(x) = H(x, \theta(x))$ where θ is in a certain Sobolev space and H is a smooth function in all of its arguments. We show that f is in the same Sobolev space and derive a bound on its Sobolev norm. The need for such a result arose in the study of a discrete approximation to a nonlinear differential equation that arose in Cox and O'Sullivan [2]. See the discussion of the examples near the end of §2.1 of that paper. We would expect that such a result could have other applications in the study of nonlinear differential equations.

We now give some definitions and notation. For a normed linear space V , let $\|v\|_V$ denote the norm on V , and let $S(r, V) = \{v \in V : \|v\|_V \leq r\}$ denote the closed ball of radius r centered at the origin. The Sobolev space of real-valued functions on a domain $\mathcal{X} \subset \mathbb{R}^d$ is denoted $W_p^s(\mathcal{X})$ as defined in Triebel [3]. Any of the various equivalent norms for $W_p^s(\mathcal{X})$ may be used. A Sobolev space of q -dimensional vector-valued functions on $\mathcal{X}$ will be denoted $W_p^s(\mathcal{X}; \mathbb{R}^q)$. We will also make use of $C^m(\mathcal{X}; \mathbb{R}^q)$ where m is an integer, which is the space of m times continuously differentiable functions equipped with an appropriate supremum norm on derivatives.

Theorem 1. *Let $\mathcal{X} \subset \mathbb{R}^d$ be a bounded, open, simply connected, nonempty set with C^∞ boundary (see [3, 3.2.1.2]). Let $d/2 < s < (d+1)/2$, and let $H: \mathcal{X} \times \mathbb{R}^q \rightarrow \mathbb{R}$ be such that for all $R > 0$, $H \in C^{k+1}(\mathcal{X} \times S(R, \mathbb{R}^q))$, where k is the largest integer less than s . If $\theta \in W_2^s(\mathcal{X}; \mathbb{R}^q)$ and $\sup_{x \in \mathcal{X}} |\theta(x)| \leq R_0$,*

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then $H(\cdot, \theta(\cdot)) \in W_2^s(\mathcal{X})$ and $\|H(\cdot, \theta(\cdot))\|_{W_2^s(\mathcal{X})} \leq M[1 + \|\theta\|_{W_2^s}]\|$ where M is a constant depending on $\mathcal{X}$, H , s , and R_0 .

Remark. Note that $\theta \in W_2^s$ implies by Sobolev's embedding theorem [1, Theorem 5.4] (noting that our assumption of C^∞ boundary for $\mathcal{X}$ implies any of the cone conditions) that $\sup_{\mathcal{X}} |\theta| < \infty$.

Proof. Note that s is not an integer. We will assume for now that $d > 1$. The proof when $d = 1$ is easier and will be indicated at the end. In this proof, we will write $M(a, b, \dots)$ for a generic constant M depending only on the quantities $a, b, \dots$. Note that dependence on $\mathcal{X}$ subsumes dependence on d and dependence on s subsumes dependence on k .

Using equation (1) of Theorem 4.4.2.1 of [3], a norm for the Besov space $B_{pq}^s(\mathcal{X})$ ($1 < p, q < \infty, 0 < r < \infty, r$ not an integer) is

(1)

$$\|f\|_{B_{pq}^r(\mathcal{X})} = \|f\|_{L_p(\mathcal{X})} + \sum_{j=1}^d \left\{ \int_{|h| \leq \delta} |h|^{-q(r-k)-d} \times \left[\int_{\mathcal{X}(h, \varepsilon, \tau)} \left| \Delta_h \frac{\partial^k f}{\partial x_j^k}(x) \right|^p dx \right]^{q/p} dh \right\}^{1/q}$$

for any $\delta, \varepsilon, \tau$ sufficiently small, and k the largest integer less than r . Here, $\Delta_h f(x) = f(x+h) - f(x)$ [3, 2.5.1, p. 189] and

$$\mathcal{X}(h, \varepsilon, \tau) = (\mathcal{X} \setminus \mathcal{X}^\tau) \cup \{z + (0, \tau]\nu_z : z \in \partial\mathcal{X} \text{ and } 0 \leq \angle(h, \nu_z) < \varepsilon\},$$

$$\mathcal{X}^\tau = \left\{ x \in \mathcal{X} : \inf_{z \in \partial\mathcal{X}} |x - z| < \tau \right\},$$

$$\nu_z = \text{inner normal to } \partial\mathcal{X} \text{ at } z,$$

$$\angle(h, \nu) = \text{angle between } h, \nu \text{ (always positive).}$$

Since

$$(2) \quad B_{pp}^s \cong W_p^s,$$

this gives an equivalent norm for the Sobolev spaces as well [3, Definition 4.2.1.1]. Here, $\cong$ denotes equality of two normed linear spaces up to equivalence of norms.

In order to denote partial derivatives of H , we will write the independent variables as $H(x, t)$, $x \in \mathcal{X}$ and $t \in \mathbb{R}^q$. Let k be as in the statement of the theorem. For fixed $\theta \in W_2^s(\mathcal{X})$ let $f(x) = H(x, \theta(x))$. Using the chain rule and product rule we have

$$(3) \quad \frac{\partial^k f}{\partial x_j^k}(x) = \frac{\partial^k H}{\partial x_j^k}(x, \theta(x)) + \sum_l c_l \frac{\partial^{k_{0l}} H}{\partial t_l^{k_{0l}}}(x, \theta(x)) \prod_i \frac{\partial^{k_{il}} \theta_l}{\partial x_j^{k_{il}}}(x)$$

where the c_l 's are constants depending on k , $k_{0l} \leq k$, all $k_{il} \geq 1$, and

$$(4) \quad \sum_{i \geq 1} k_{il} = k.$$

We do not indicate the upper limits on the indices l and i in (3) (the upper limit on i depends on l). Suffice it to note that they are finite and depend only

on k and q . We have indicated through the subscripts that the k_{il} depend on l , but we will drop that dependence for convenience henceforth. We will assume for now that there is more than one summand in (4). The case of one summand will be easy to treat.

Using the hypothesis $H \in C^{k+1}(\mathcal{X} \times S(R_0, \mathbb{R}^q))$, it is easy to check that

$$(5) \quad \int_{|h| \leq \delta} |h|^{-2(s-k)-d} \int_{\mathcal{X}(h, \varepsilon, \tau)} \left[\frac{\partial H}{\partial x_j}(x+h, \theta(x+h)) - \frac{\partial H}{\partial x_j}(x, \theta(x)) \right]^2 dx dh \leq M(H, R_0) < \infty.$$

Using boundedness of $\partial^{k_0} H / \partial t_l^{k_0}$ and Hölder's inequality, if m is a fixed index, with the same range as i , then

$$(6) \quad \begin{aligned} & \int_{|h| \leq \delta} |h|^{-2(s-k)-d} \int_{\mathcal{X}(h, \varepsilon, \tau)} \left\{ \frac{\partial^{k_0} H}{\partial t_l^{k_0}}(x+h, \theta(x+h)) \left[\prod_{i < m} \frac{\partial^{k_i} \theta_l}{\partial x_j^{k_i}}(x+h) \right] \right. \\ & \quad \times \left. \left[\frac{\partial^{k_m} \theta_l}{\partial x_j^{k_m}}(x+h) - \frac{\partial^{k_m} \theta_l}{\partial x_j^{k_m}}(x) \right] \left[\prod_{i > m} \frac{\partial^{k_i} \theta_l}{\partial x_j^{k_i}}(x) \right] \right\}^2 dx dh \\ & \leq M(\mathcal{X}, H, R_0) \left\{ \prod_{i \neq m} \left[\int_{\mathcal{X}} \left| \frac{\partial^{k_i} \theta_l}{\partial x_j^{k_i}}(x) \right|^{2q_i} dx \right]^{1/q_i} \right\} \\ & \times \int_{|h| \leq \delta} |h|^{-2(s-k)-d} \left\{ \int_{\mathcal{X}(h, \varepsilon, \tau)} \left| \frac{\partial^{k_m} \theta_l}{\partial x_j^{k_m}}(x+h) - \frac{\partial^{k_m} \theta_l}{\partial x_j^{k_m}}(x) \right|^{2q_m} dx \right\}^{1/q_m} dh \end{aligned}$$

provided

$$(7) \quad \sum_{i \geq 1} 1/q_i \leq 1.$$

(Here, the q_i depend on l and m , but this will not be important, so we do not indicate this dependence.) It is necessary to check that each of the integrals on the r.h.s. of (6) is finite. By the embedding Theorem 4.6.1(c) [3, pp. 327–328]

$$(8) \quad B_{p_1, q}^{r_1} \subset B_{p_2, q}^{r_2}$$

provided

$$(9) \quad 1 < p_1 \leq p_2 < \infty$$

and

$$(10) \quad r_1 - d/p_1 \geq r_2 - d/p_2.$$

In (8), the embedding operator (inclusion map) is continuous, which means the space on the l.h.s. has a stronger norm than the space on the r.h.s. Furthermore, by part (a) of the aforementioned theorem

$$(11) \quad B_{q, p_1}^{r_1} \subset B_{q, p_2}^{r_1}$$

provided

$$(12) \quad p_1 \leq p_2,$$

again with continuous embedding. We will use these results to show that the q_i can be chosen satisfying (7) so that all the integrals on the r.h.s. of (6) are finite.

Now

$$(13) \quad \int_{\mathcal{X}} \left| \frac{\partial^{k_i} \theta_l}{\partial x_j^{k_i}}(x) \right|^{2q_i} dx \leq [M(s, \mathcal{X}) \|\theta|W_2^s\|]^2 < \infty$$

provided $q_i > 1$ and

$$k_i - d/2q_i \leq s - d/2$$

(use $W_{2q_i}^{k_i} = B_{2q_i, 2q_i}^{k_i} \supset B_{2q_i, 2}^{k_i} \supset B_{22}^s = W_2^s$, which follow from (8) through (12), and the fact that $W_{2q_i}^{k_i}$ norm squared dominates the integral in (13)), which is equivalent to

$$(14) \quad 1 < q_i \leq d/[2(k_i - s) + d].$$

Note that $k_i - s < 0$ since $k_i \leq k < s$ and that $d + 2(k_i - s) > 0$ since $k_i \geq 1$ and $s < d/2 + 1$, so the last member of (14) is greater than 1. Also,

$$(15) \quad \int_{|h| \leq \delta} |h|^{-2(s-k)-d} \left\{ \int_{\mathcal{X}(h, \varepsilon, \tau)} \left| \frac{\partial^{k_m} \theta_l}{\partial x_j^{k_m}}(x+h) - \frac{\partial^{k_m} \theta_l}{\partial x_j^{k_m}}(x) \right|^{2q_m} dx \right\}^{1/q_m} dh \\ \leq [M(s, \mathcal{X}) \|\theta|W_2^s\|]^2 < \infty$$

provided $q_m > 1$ and

$$k_m + (s - k) - d/2q_m \leq s - d/2$$

(use $B_{2q_m, 2}^{k_m+(s-k)} \supset B_{22}^s = W_2^s$, and the equivalent norm in (1)), which is equivalent to

$$(16) \quad 1 < q_m \leq d/[2(k_m - k) + d].$$

Note that $k_m < k$ since there is more than 1 summand in (4), so the denominator in the last member of (16) is less than d , and it is positive since $k_m \geq 1$ and $k \leq d/2$, so the last member of (16) is greater than 1. (Note that if there is only 1 summand in (4) then finiteness of (6) is trivial since it is then one of the terms in the equivalent norm (1) for $\|\theta|W_2^s\|$.) To finish the proof that all integrals on the r.h.s. of (6) are finite and, in fact, that the whole r.h.s. of (6) is dominated by a constant (depending only on s , $\mathcal{X}$, H , and R_0) multiple of $\|\theta|W_2^s\|^2$, it suffices to show we can find q_i 's satisfying (7), (14), and (16), which is possible if

$$1 \geq [d - 2(k - k_m)]/d + \sum_{1 \leq i \neq m} [2(k_i - s) + d]/d.$$

Now the r.h.s. is equal to

$$(2/d) \left[\sum_i k_i - (s - d/2) \sum_i 1 + (s - k) \right] \\ = (2/d) \left[k - (s - d/2) \sum_i 1 + (s - k) \right] \\ = (2/d) \left[s - (s - d/2) \sum_i 1 \right] \leq (2/d) [s - (s - d/2)] \leq 1,$$

so we may find q_i 's such that (6) is bounded by a constant (depending on s , $\mathcal{X}$, H , and R_0) multiple of $\|\theta\|_{W_2^s}$.

Finally, we need to consider integrals of the form

$$\begin{aligned}
 & \int_{|h| \leq \delta} |h|^{-2(s-k)-d} \int_{\mathcal{X}(h, \varepsilon, \tau)} \left\{ \left[\Delta_h \frac{\partial^{k_0} H}{\partial t_l^{k_0}}(x, \theta(x)) \right] \prod_{i \geq 1} \frac{\partial^{k_i} \theta_l}{\partial x_j^{k_i}}(x) \right\}^2 dx dh \\
 (17) \quad & \leq \int_{|h| \leq \delta} |h|^{-2(s-k)-d} \left\{ \int_{\mathcal{X}(h, \varepsilon, \tau)} \left| \Delta_h \frac{\partial^{k_0} H}{\partial t_l^{k_0}}(x, \theta(x)) \right|^{2q_0} dx \right\}^{1/q_0} dh \\
 & \quad \times \prod_{i \geq 1} \left\{ \int_{\mathcal{X}} \left| \frac{\partial^{k_i} \theta_l}{\partial x_j^{k_i}}(x) \right|^{2q_i} dx \right\}^{1/q_i}
 \end{aligned}$$

where

$$(18) \quad 1/q_0 + \sum_{i \geq 1} 1/q_i \leq 1$$

and (14) holds for all $i \geq 1$, which implies that the product of integrals over $i \geq 1$ on the r.h.s. of (17) is finite. For the other factor, apply the mean value theorem to show it is equal to

$$\begin{aligned}
 (19) \quad & \int_{|h| \leq \delta} |h|^{-2(s-k)-d} \left\{ \int_{\mathcal{X}(h, \varepsilon, \tau)} \left| \sum_{j=1}^d \frac{\partial^{k_0+1} H}{\partial x_j \partial t_l^{k_0}}(x + \alpha h, \alpha \theta(x + h) + (1 - \alpha)\theta(x)) h_j \right. \right. \\
 & \quad \left. \left. + \sum_{j=1}^q \frac{\partial^{k_0+1} H}{\partial t_j \partial t_l^{k_0}}(x + \alpha h, \alpha \theta(x + h) + (1 - \alpha)\theta(x)) \Delta_h \theta_j(x) \right|^{2q_0} dx \right\}^{1/q_0} dh
 \end{aligned}$$

where $\alpha = \alpha(x, \theta(x), h)$ is between 0 and 1. Since $\partial^{k_0+1} H / \partial x_j \partial t_l^{k_0}$ is bounded and $|h_j| \leq |h|$, it is easy to see that the first summation above after the integrations is bounded by a constant depending on s , $\mathcal{X}$, H , and R_0 . For the second summation, we may use boundedness of $\partial^{k_0+1} H / \partial t_j \partial t_l^{k_0}$ to pull that factor out, and then we must deal with terms of the form

$$(20) \quad \int_{|h| \leq \delta} |h|^{-2(s-k)-d} \left\{ \int_{\mathcal{X}(h, \varepsilon, \tau)} |\Delta_h \theta_j(x)|^{2q_0} dx \right\}^{1/q_0} dh \leq \|\theta_j\|_{B_{2q_0, 2}^{(s-k)}}^2$$

where the latter inequality follows from (1). Now $B_{2q_0, 2}^{(s-k)} \supset B_{22}^s$ if $1 < q_0$ and

$$(s - k) - d/2q_0 \leq s - d/2,$$

which is equivalent to

$$(21) \quad 1 < q_0 \leq d/[d - 2k].$$

Since $d > 1$, this latter pair of inequalities is possible. We also need from (18) that

$$(22) \quad 1/q_0 \leq 1 - (2/d) \left[k - (s - d/2) \sum_{i \geq 1} 1 \right].$$

One can check that (21) and (22) are compatible, so we can find the q_i 's with the requisite properties.

Since the r.h.s. of (1) can be bounded by a linear combination of terms of the form of (17), (6), (5), and $\|H\|C(\mathcal{X} \times S(R_0, \mathbb{R}^q))\|$ (for the first term on the r.h.s. of (1)), the result follows.

Finally we briefly indicate the $d = 1$ case. Here, $k = 0$, so no chain rule is involved. One uses the mean value theorem in (1) similar to the last step above but then proceeds directly to $\|\theta\|W_2^s\|$ since no Hölder step is used. This completes the proof. $\square$

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DEPARTMENT OF STATISTICS, RICE UNIVERSITY, P. O. BOX 1892, HOUSTON, TEXAS 77251

E-mail address: dcox@rice.edu

DEPARTMENTS OF STATISTICS AND BIostatISTICS, UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON 98195

E-mail address: finbarr@stat.washington.edu