A COMPARISON BETWEEN EULER AND CESÀRO METHODS OF SUMMABILITY

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ABSTRACT. It is well known that there are sequences that are summable by every Cesàro method C_r ($r > 0$) but are not summable by any Euler method E_p ($0 < p < 1$). It is proved here that on the other hand there are sequences that are summable by every Euler method E_p ($0 < p < 1$) but are not summable by any Cesàro method.

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We consider the regular Cesàro methods C_r ($r > 0$) and the regular Euler methods E_p ($0 < p < 1$), where the latter is defined as in [5]; that is, the E_p -transform of any sequence $\{x_k\}$ is given by $\{t_n(p)\}$ where

$$(1) \quad t_n(p) = \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k} x_k \quad (n = 0, 1, \dots).$$

Let (C_r) and (E_p) denote the summability fields of C_r and E_p respectively. It is well known that

$$(E_p) \not\subset (C_r) \quad \text{and} \quad (C_r) \not\subset (E_p), \quad 0 < p < 1, \quad r > 0$$

(see, for instance, [5, Satz 64.V]). It is also well known that if $0 < p < 1$ and $r > 0$,

$$m \cap \bigcup_{0 < q < 1} (E_q) = m \cap (E_p) \subset m \cap (C_r) = m \cap \bigcup_{s > 0} (C_s),$$

where m is the set of all bounded sequences. Further, it is known that

$$\bigcap_{r > 0} (C_r) \not\subset \bigcup_{0 < p < 1} (E_p);$$

that is, there exist sequences x that are summable by every Cesàro method C_r ($r > 0$) but not by any Euler method E_p [2, p. 251; 1, p. 213]. Such a sequence x can be made to satisfy certain additional conditions too, like being bounded,

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or unbounded, or that the series $\sum a_n = \sum (x_n - x_{n-1})$ have gaps of a certain type (see [3]). But the question whether

$$\bigcap_{0 < p < 1} (E_p) \subset (C_r) \quad \text{for some } r > 0$$

seems to have remained open. The object of the present paper is to prove that the answer to the question is in the negative. In fact, we prove rather more.

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Theorem. (a) *There exist sequences that are E_p -summable for every $0 < p < 1$ but are not summable by any Cesàro method; that is,*

$$\bigcap_{0 < p < 1} (E_p) \not\subset \bigcup_{r > 0} (C_r).$$

(b) *There exists a sequence $\{V^{(m)}\}$ of row-finite regular matrix methods such that*

$$(V^{(1)}) \not\subset \bigcup_{r > 0} (C_r) \quad \text{and} \quad (V^{(1)}) \subset (V^{(2)}) \subset \cdots \subset \bigcap_{0 < p < 1} (E_p).$$

Proof. (a) Let the sequence $\{x_n\}$ be defined by

$$(2) \quad x_k = (-1)^k \sum_{r=0}^k c_r [k]_r \quad (k = 0, 1, \dots)$$

where $[k]_r := k(k-1)\cdots(k-r+1)$ $\{r \text{ factors}\}$ (and is interpreted as 0 when $r = 0$ or $k < r$) and where $\{c_r\}$ is some fixed sequence of positive numbers which satisfy a certain condition to be imposed later. For $0 < p < 1$, we have for each $n = 0, 1, \dots$,

$$\begin{aligned} t_n(p) &= \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k} (-1)^k \sum_{r=0}^k c_r [k]_r, \\ (3) \quad &= \sum_{k=0}^n c_r \sum_{k=r}^n \binom{n}{r} p^k (1-p)^{n-k} (-1)^k [k]_r, \\ &= \sum_{r=0}^n c_r \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k} (-1)^k [k]_r, \end{aligned}$$

since $[k]_r = 0$ for $k < r$. Now

$$(4) \quad \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k} x^k = [px + (1-p)]^n.$$

Differentiating each side of this expression r times (with respect to x) and then putting $x = -1$, we see that the inner sum in (3) is equal to

$$(-1)^r [n]_r p^r (1-2p)^{n-r}.$$

Hence

$$(5) \quad t_n(p) = \sum_{r=0}^n c_r (-1)^r [n]_r p^r (1-2p)^{n-r} = \sum_{r=0}^n c_r a_{nr}(p), \quad \text{say.}$$

Now let $\{a_r\}$, $\{b_r\}$ be (fixed) sequences of positive numbers decreasing to 0, with $a_0 < 1/2$, and let $\sum b_r < \infty$. We note that for a given r , $a_{nr}(p) \rightarrow 0$ as $n \rightarrow \infty$, uniformly for p in any closed interval contained in $(0, 1)$. It is bounded for any fixed n ; hence it is bounded for all n , uniformly in p for p in any such interval. Hence we can choose $\{c_r\}$ so that for each r ,

$$c_r |a_{nr}(p)| < b_r \quad (a_r \leq p \leq 1 - a_r, \text{ for all } n).$$

Now consider an arbitrary fixed value of p , with $0 < p < 1$, and let $\varepsilon > 0$ be arbitrarily given. Since $a_r \rightarrow 0$, we see that $p \in [a_r, 1 - a_r]$ for sufficiently large r . Choose R large enough for this to hold for all $r \geq R$ such that, further, $\sum_{r=R}^{\infty} b_r < \varepsilon$. Then for all n ,

$$(6) \quad \sum_{r=R}^{\infty} c_r |a_{nr}(p)| < \varepsilon.$$

But, for fixed r and p , we have $a_{nr}(p) \rightarrow 0$ as $n \rightarrow \infty$; so, having fixed R , we have

$$\sum_{r=0}^{R-1} c_r |a_{nr}(p)| < \varepsilon \quad \text{for all } n \text{ sufficiently large.}$$

Then it follows from (5) and (6) that $t_n(p) \rightarrow 0$ as $n \rightarrow \infty$. Thus the sequence $x = \{x_n\}$ is summable by the method E_p , for arbitrary $p \in (0, 1)$; that is, $x \in \bigcap_{0 < p < 1} (E_p)$.

On the other hand, for each fixed positive integer r ,

$$|x_k| \geq c_r [k]_r = c_r k(k-1) \cdots (k-r+1) \geq (1/2) c_r k^r$$

if k is sufficiently large. Hence $\{x_n\}$ does not satisfy the conditions of the limitation theorem for summability by the Cesàro method C_r [1, pp. 101-102; 5, p. 104]. Since this is true for every positive integer r , the sequence $\{x_n\}$ is not summable by C_r for any real value of $r > 0$. This proves part (a) of the theorem.

(b) We note that all of the above remains valid if, for each $r \geq 0$, we replace c_r by any smaller positive number c'_r . Hence, if we define the sequences $x^{(m)}$ ($m = 1, 2, \dots$) by setting

$$x_k^{(m)} = (-1)^k \sum_{r=0}^k c_r \left(\frac{r}{r+m} \right) [k]_r \quad (k = 0, 1, \dots),$$

then $x^{(m)} \in \bigcap_{0 < p < 1} (E_p)$, but $x^{(m)} \notin \bigcup_{r > 0} (C_r)$ ($m = 1, 2, \dots$).

Now let $y = \{y_k\} = a_1 x^{(1)} + \dots + a_s x^{(s)}$ be an arbitrary given linear combination of the sequences $x^{(1)}, x^{(2)}, \dots$, where not all the a_j are zero. Then for each $k = 0, 1, \dots$,

$$(7) \quad y_k = (-1)^k \sum_{r=0}^k c_r [k]_r \left(\sum_{j=1}^s \frac{r a_j}{r+j} \right) = (-1)^k \sum_{r=0}^k c_r [k]_r A(r)$$

where $A(x) = \sum_{j=1}^s x a_j / (x+j)$ ($x \geq 0$). Now every (nonnegative) solution of the equation $A(x) = 0$ is a solution of the equation $P(x) = 0$, where $P(x) = \sum_{j=1}^s x a_j \prod_{1 \leq i \leq s, i \neq j} (x+i)$. If the polynomial $P(x)$ were identically zero, then

$P(-j) = 0$ and hence $a_j = 0$ for $1 \leq j \leq s$, contrary to the assumption about the numbers a_j . Hence $P(x)$ has only a finite number of zeros, so there exists a positive integer N such that $P(x)$, and therefore also $A(x)$, is nonzero and of constant sign in the interval $[N, \infty)$. From the equation (7), we have for each $k > N$,

$$y_k = (-1)^k \sum_{r=0}^{N-1} c_r[k]_r A(r) + (-1)^k \sum_{r=N}^k c_r[k]_r A(r) = S_k + T_k \quad (\text{say}).$$

Then S_k is the sum of N terms, each of which is $O(k^{N-1})$ as $k \rightarrow \infty$, and hence $S_k = O(k^{N-1})$. On the other hand, since $A(r)$ is nonzero and has constant sign for all $r \geq N$ (and since $c_r > 0$ for all r), we see that

$$\begin{aligned} |T_k| &\geq c_N[k]_N |A(N)| = c_N k(k-1)(k-2) \cdots (k-N+1) |A(N)| \\ &\geq (1/2) c_N |A(N)| k^N \quad \text{for all sufficiently large } k. \end{aligned}$$

Hence $T_k \neq O(k^{N-1})$ as $k \rightarrow \infty$. Since $S_k = O(k^{N-1})$, it follows that $y_k = S_k + T_k \neq O(1)$. That is, no nontrivial linear combination of the sequences $x^{(1)}, x^{(2)}, \dots$ yields a bounded sequence. It follows then from Theorem 3 of [4] that for each $m \geq 1$ there exists a regular row-finite matrix method $V^{(m)}$ whose summability field is the smallest sequence space containing c (the convergent sequences) and the sequences $x^{(1)}, x^{(2)}, \dots, x^{(m)}$. Then we have $(V^{(1)}) \not\subset \bigcup_{r>0} (C_r)$ and $(V^{(1)}) \subset (V^{(2)}) \subset \cdots \subset \bigcap_{0 < p < 1} (E_p)$. This completes the proof of the theorem.

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