EVERY UNCOUNTABLE ABELIAN GROUP ADMITS A NONNORMAL GROUP TOPOLOGY

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ABSTRACT. If G is a locally compact Abelian group, let G^+ denote the underlying group of G equipped with the weakest topology that makes all the continuous characters of G continuous. Thus defined, G^+ is a totally bounded topological group. We prove:

Theorem. G^+ is normal if and only if G is σ -compact.

When G is discrete, this theorem answers in the negative a question posed in 1990 by E. van Douwen, and it partially solves a problem posed in 1945 by A. Markov.

INTRODUCTION

All spaces are completely regular and Hausdorff, and all groups are Abelian.

For a locally compact Abelian group G define G^+ as in the abstract. The topology of G^+ is known as the *weak topology on G* and has been studied, among others, by Glicksberg in [4] and by Trigos-Arrieta in [10, 11]. In his 1990 paper E. van Douwen studied the space G^+ for G discrete, i.e., G equipped with its *maximal totally bounded (topological group) topology*, and asked whether this topology was normal when $|G| > \omega$.

On the other hand, A. Markov, back in 1945, asked whether every uncountable group—Abelian or not—would admit a nonnormal Hausdorff topological group topology. Without offering any details or citing the existence of a manuscript in preparation, Markov attributed to E. Livenson a proof that every uncountable *Abelian* group admits such a topology. Despite the effort of many mathematicians, it has been impossible to determine the basis of this attribution and the nature of Levinson's presumed construction. It is clear that by considering the maximal totally bounded topology, the negative answer to van Douwen's question provides a positive answer to Markov's problem in the Abelian situation. Those interested in the nonabelian case are referred to the informative survey [1].

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THE RESULT

Our strategy is to deal first with the situation when G is discrete, leaving the general case as a corollary.

A subset Λ of an Abelian group G is said to be *independent* if $0 \notin \Lambda$, and for every choice $\{a_i\}_{i=1}^n$ from Λ and $\{m_i\}_{i=1}^n$ of integers, the equality $m_1 a_1 + \cdots + m_n a_n = 0$ implies $m_i a_i = 0$ for all i . See [3, §16].

As noticed in [5, (1.4)], independent subsets of G are always closed and discrete in G^+ .

If S is a subset of a space X , we denote by $\text{Cl}_X S$ the closure of S in X . If S is a subset of a group G , we denote by $\langle S \rangle$ the subgroup of G generated by S . βX and $\mathbf{b}G$ will denote the *Čech-Stone compactification* of X , and the *Bohr compactification* of G , respectively. The main result of this article follows. The style of proof here is due to the referee, based on the author's arguments; see Trigós-Arrieta [12].

Theorem. *If G is a (discrete) Abelian group, then $|G| > \omega$ if and only if G^+ is not normal.*

Proof. $(\Rightarrow)$ Let G be an uncountable group. By [3, §16] or [7] we can take an uncountable independent subset Λ of G . By [5, (1.4)] and [2, (1.13(a))], we can assume that any subset of Λ is C^* -embedded in $\mathbf{b}G$. Divide Λ into two subsets Λ_1 and Λ_2 of the same cardinality, and let $H_1 = \langle \Lambda_1 \rangle$ and $H_2 = \langle \Lambda_2 \rangle$. Clearly $\text{Cl}_{\mathbf{b}H_1} \Lambda_1 = \beta \Lambda_1$ and $\text{Cl}_{\mathbf{b}H_2} \Lambda_2 = \beta \Lambda_2$. This implies that $(\beta \Lambda)^2 \subseteq \mathbf{b}H_1 \times \mathbf{b}H_2 \subseteq \mathbf{b}G$. If $\Delta := \{(\lambda, \lambda) : \lambda \in \Lambda\}$, we note as well that Δ and $\Phi := \Lambda^2 \setminus \Delta$ are closed and discrete subsets of G^+ with $\Delta \cap \Phi = \emptyset$. One must show that Δ and Φ cannot be separated by open sets in G^+ . Let U be an arbitrary open set in G^+ containing Φ . By [9, 1.5 and 1.6] $F := \text{Cl}_{\mathbf{b}G} U$ is a zero-set of $\mathbf{b}G$. Since $(\beta \Lambda)^2 \setminus F$ is Lindelöf (being cozero in $(\beta \Lambda)^2$) and Δ is discrete and open in $(\beta \Lambda)^2$, it follows (by considering for each $p \in (\beta \Lambda)^2 \setminus F$ an open (in $(\beta \Lambda)^2$) rectangle $R_p \subseteq (\beta \Lambda)^2 \setminus F$, if necessary) that $(\beta \Lambda)^2 \setminus F$ is at most countable. Therefore, $\Delta \cap F \neq \emptyset$, implying that $\Delta \cap \text{Cl}_{G^+} U \neq \emptyset$. If V were an open set in G^+ such that $\Delta \subseteq V$ and $U \cap V = \emptyset$, we would have $\Delta \cap \text{Cl}_{G^+} U = \emptyset$. This contradiction proves $(\Rightarrow)$.

$(\Leftarrow)$ Obvious. $\square$

For the general case we can proceed as follows: Clearly the weak topology on a locally compact σ -compact Abelian group must be normal. Let G be a locally compact Abelian group which is not σ -compact. By [6, (24.30)] G is topologically isomorphic to $\mathbf{R}^n \times G_0$, where $\mathbf{R}$ denotes the real line, n is a nonnegative integer, and G_0 has a compact open subgroup H_0 . Apply the theorem to the uncountable discrete group G_0/H_0 and use [10, (2.2); 11, §3; 6, (5.18)]. Thus we have

Corollary. *If G is a locally compact Abelian group, then G is σ -compact if and only if G^+ is normal.*

Remark. The paper [12] deals with the same results of this article in greater detail.

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