A REMARK ON COINCIDENCE THEOREMS

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ABSTRACT. Using a different approach, we prove a general coincidence theorem of multivalued mappings which have contractible values in non-convex setting.

Let $\mathbb{N}$ and $\mathbb{R}$ denote the set of all natural numbers and the set of all real numbers, respectively. For each $n \in \mathbb{N}$, let $N = \{0, 1, \dots, n\}$ and $\Delta_N = \text{co}\{e_0, \dots, e_n\}$ be the standard simplex of dimension n , where $\{e_0, \dots, e_n\}$ is the canonical basis of $\mathbb{R}^{n+1}$. For $J \in \mathcal{F}(N)$, $\Delta_J = \text{co}\{e_j : j \in J\}$, where $\mathcal{F}(N)$ denotes the family of all non-empty subsets of N . A topological space X is said to be *contractible* if the identity mapping I_X of X is homotopic to a constant function. Given a non-empty set X , let 2^X denote the family of all non-empty subsets of X .

In 1986, by using Browder's selection theorem (e.g., see [2]) for the multivalued mapping with open inverse values, Komiya [10] gave the following coincidence theorem:

Theorem (Komiya). *Let X be a non-empty convex subset of a Hausdorff topological vector space E , and let Y be a non-empty compact convex subset of a Hausdorff topological vector space W . Suppose $A : X \rightarrow 2^Y$ is upper semicontinuous with closed and convex values and $B : Y \rightarrow 2^X$ has convex values such that $B^{-1}(x)$ is open in Y for each $x \in X$. Then there exists $(x_0, y_0) \in X \times Y$ such that $x_0 \in B(y_0)$ and $y_0 \in A(x_0)$.*

In this note, we shall give a very general form of the above coincidence theorem in a non-convex setting by using a different but simple approach.

The following is Theorem 1 of Horvath [7] (see also Theorem 1 of [8]):

Lemma 1. *Let X be a topological space. For each non-empty subset J of $N = \{0, 1, \dots, n\}$, let F_J be a non-empty contractible subset of X with $F_J \subset F_{J'}$ whenever $\emptyset \neq J \subset J' \subset \{0, 1, \dots, n\}$. Then there exists a continuous function $f : \Delta_N \rightarrow X$ such that $f(\Delta_J) \subset F_J$ for each non-empty subset J of $\{0, 1, \dots, n\}$.*

By employing the same arguments of Eilenberg and Montgomery [3, pp. 216–217] and using Theorem 6.3 of Górniewicz [4, p. 111] instead of the coincidence

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theorem used in [3, pp. 216–217], the following lemma was obtained by Shioji [11, p. 188]:

Lemma 2 (Eilenberg and Montgomery, Górniewicz, and Shioji). *Let Δ_N be an n -dimensional simplex with the Euclidean topology, and let Y be a compact topological space. Let $\psi : Y \rightarrow \Delta_N$ be a single-valued continuous map, and let $T : \Delta_N \rightarrow 2^Y$ be a set-valued upper semicontinuous mapping with non-empty compact contractible values. Then there exists $x_0 \in \Delta_N$ such that $x_0 \in \psi \circ T(x_0)$, where $\psi \circ T$ denotes the composition of mapping T with ψ .*

Now we have the following:

Theorem 1. *Let X be a contractible space and Y a Hausdorff compact space. Let $A : X \rightarrow 2^Y$ be upper semicontinuous with non-empty compact contractible values. Suppose that $B : Y \rightarrow 2^X$ is such that*

- (a) $B^{-1}(x)$ is open for each $x \in X$ and
- (b) for each open set S in Y , the set $\cap_{y \in S} B(y)$ is empty or contractible.

Then there exist $w_0 \in X$ and $z_0 \in Y$ such that $w_0 \in B(z_0)$ and $z_0 \in A(w_0)$.

Proof. We first show that there exist an n -simplex Δ_N and two functions $f : \Delta_N \rightarrow X$ and $\psi : Y \rightarrow \Delta_N$ such that $f(\psi(y)) \in B(y)$ for all $y \in Y$.

Since Y is compact, by (a), there exists a finite subset $\{x_0, \dots, x_n\}$ of X such that $Y = \cup_{i=0}^n B^{-1}(x_i)$. Now for each non-empty subset J of $N := \{0, \dots, n\}$, we define

$$F_J = \begin{cases} \cap \{B(y) : y \in \cap_{j \in J} B^{-1}(x_j)\}, & \text{if } \cap_{j \in J} B^{-1}(x_j) \neq \emptyset, \\ X, & \text{otherwise.} \end{cases}$$

Note that if $y \in \cap_{j \in J} B^{-1}(x_j)$, then $\{x_j : j \in J\} \subset B(y)$. If $\cap_{j \in J} B^{-1}(x_j) \neq \emptyset$, then $F_J = \cap \{B(y) : y \in \cap_{j \in J} B^{-1}(x_j)\}$ is non-empty and contractible. It is clear that $F_J \subset F_{J'}$ whenever $\emptyset \neq J \subset J' \subset \{0, \dots, n\}$. Thus F satisfies all hypotheses of Lemma 1. By Lemma 1, there is a continuous function $f : \Delta_N \rightarrow X$ such that $f(\Delta_J) \subset F_J$ for all $J \in \mathcal{F}(N)$. Let $\{\psi_i : i \in N\}$ be a continuous partition of unity subordinated to the covering $\{B^{-1}(x_i) : i \in N\}$, i.e., for each $i \in N$, $\psi_i : Y \rightarrow [0, 1]$ is continuous, $\{y \in Y : \psi_i(y) \neq 0\} \subset B^{-1}(x_i)$ such that $\sum_{i=0}^n \psi_i(y) = 1$ for all $y \in Y$. Define $\psi : Y \rightarrow \Delta_N$ by $\psi(y) = (\psi_0(y), \psi_1(y), \dots, \psi_n(y))$ for each $y \in Y$. Then $\psi(y) \in \Delta_{J(y)}$ for all $y \in Y$, where $J(y) = \{i \in \{0, \dots, n\} : \psi_i(y) \neq 0\}$. Therefore $f(\psi(y)) \in f(\Delta_{J(y)}) \subset F_{J(y)} \subset B(y)$. Since A is upper semicontinuous with non-empty compact and contractible values and f is continuous, it follows that the composition $A \circ f : \Delta_N \rightarrow 2^Y$ is also upper semicontinuous with non-empty compact and contractible values. Note that $\psi : Y \rightarrow \Delta_N$ is continuous. By Lemma 2 above, there exists $x_0 \in \Delta_N$ such that $x_0 \in \psi \circ (A \circ f(x_0))$. Let $w_0 := f(x_0)$. Then $w_0 = f(x_0) \in f \circ (\psi \circ (A \circ f(x_0))) = f \circ (\psi \circ (A(w_0)))$ so that there exists $z_0 \in A(w_0)$ such that $w_0 = f \circ \psi(z_0) \in B(z_0)$. $\square$

Since each convex set in Hausdorff topological vector spaces is contractible, clearly Komiya's theorem is a special case of Theorem 1. Recently, by the generalization of the classic Knaster-Kuratowski-Mazurkiewicz theorem, Horvath [7] gave a number of coincidence theorems in which both mappings involve the property of open inverse values (or open image values); for example, see Theorem 3 and Corollaries 3–6 in [7]. Note that the mapping A in Theorem 1 is upper semicontinuous, so that our Theorem 1 is not comparable with

those coincidence results given by Horvath [7]. For more applications of coincidence theorems, we refer to Aubin [1], Browder [2], Granas and Liu [5], Ha [6], Komiya [10], Yuan [12], and the references therein.

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