SINGULAR PERTURBATIONS OF INTEGRODIFFERENTIAL EQUATIONS IN BANACH SPACE

JAMES H. LIU

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ABSTRACT. Let $\varepsilon > 0$ and consider

$$\varepsilon^2 u''(t; \varepsilon) + u'(t; \varepsilon) = Au(t; \varepsilon) + \int_0^t K(t-s)Au(s; \varepsilon) ds + f(t; \varepsilon), \quad t \geq 0,$$

$$u(0; \varepsilon) = u_0(\varepsilon), \quad u'(0; \varepsilon) = u_1(\varepsilon),$$

and

$$w'(t) = Aw(t) + \int_0^t K(t-s)Aw(s) ds + f(t) \quad t \geq 0, \quad w(0) = w_0,$$

in a Banach space X when $\varepsilon \rightarrow 0$. Here A is the generator of a strongly continuous cosine family and a strongly continuous semigroup, and $K(t)$ is a bounded linear operator for $t \geq 0$. With some convergence conditions on initial data and $f(t; \varepsilon)$ and smoothness conditions on $K(\cdot)$, we prove that if $\varepsilon \rightarrow 0$, then $u(t; \varepsilon) \rightarrow w(t)$ in X uniformly for $t \in [0, T]$ for any fixed $T > 0$. We will apply this to an equation in viscoelasticity.

1. INTRODUCTION

We study integrodifferential equations

$$(1.1) \quad \begin{aligned} \varepsilon^2 u''(t; \varepsilon) + u'(t; \varepsilon) &= Au(t; \varepsilon) + \int_0^t K(t-s)Au(s; \varepsilon) ds \\ &\quad + f(t; \varepsilon), \quad t \geq 0, \\ u(0; \varepsilon) &= u_0(\varepsilon), \quad u'(0; \varepsilon) = u_1(\varepsilon), \end{aligned}$$

and

$$(1.2) \quad \begin{aligned} w'(t) &= Aw(t) + \int_0^t K(t-s)Aw(s) ds + f(t), \quad t \geq 0, \\ w(0) &= w_0, \end{aligned}$$

in a Banach space X , with A the generator of a strongly continuous cosine family and a strongly continuous semigroup, and $K(t)$ a bounded linear operator for $t \geq 0$. We regard (1.2) as the limiting equation of (1.1) as $\varepsilon \rightarrow 0$.

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Now, (1.2) is of lower order of derivative (in t); in this sense we say that we are dealing with the singular perturbation problems.

There are many studies on singular perturbations (see, e.g., Goldstein [6], Hale and Raugel [10], Smith [13], Grimmer and Liu [8], and the references therein). Since this work was influenced by Fattorini [5], we only state some results of [5].

Fattorini [5] considered the singular perturbations for

$$(1.3) \quad \begin{aligned} \varepsilon^2 u''(t; \varepsilon) + u'(t; \varepsilon) &= Au(t; \varepsilon) + f(t; \varepsilon), \quad t \geq 0, \\ u(0; \varepsilon) &= u_0(\varepsilon), \quad u'(0; \varepsilon) = u_1(\varepsilon), \end{aligned}$$

and

$$(1.4) \quad w'(t) = Aw(t) + f(t), \quad t \geq 0, \quad w(0) = w_0,$$

with A the generator of a strongly continuous cosine family and a strongly continuous semigroup in a Banach space X and proved that:

For any $T > 0$, if $f(\cdot; \varepsilon) \rightarrow f$ in $L^1([0, T], X)$ and $u_0(\varepsilon) \rightarrow w_0$, $\varepsilon^2 u_1(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, then $u(t; \varepsilon) \rightarrow w(t)$ in X uniformly for $t \in [0, T]$ as $\varepsilon \rightarrow 0$.

We will prove here with some smoothness conditions on $K(\cdot)$ that exactly the same statements as above hold for (1.1) and (1.2). The methods we will use in studying the singular perturbations for integrodifferential equations are as follows: We first use the technique introduced in [1, 2, 11, 12] to change (1.1) and (1.2) into equations that look like (1.3) and (1.4), and then estimate $u(t; \varepsilon) - w(t)$. Note that $u(\cdot; \varepsilon) - w(\cdot)$ will also appear as an integrand, so Gronwall's inequality is used to solve the problem. Finally we apply this result to an equation in viscoelasticity.

2. SINGULAR PERTURBATIONS

In this paper we make the following hypotheses:

(H1) Operator A generates a strongly continuous cosine family $C(\cdot)$ and a strongly continuous semigroup $S(\cdot)$. (See [5].)

(H2) For $t \geq 0$, $K(t)$, $K'(t)$, $K''(t) \in B(X)$ ($B(X)$ = space of all bounded linear operators on X). For $x \in X$, Kx , $K'x$, $K''x \in L^1_{\text{loc}}(R^+, X)$. Here K' , K'' are the strong derivatives.

(H3) $f(\cdot; \varepsilon)$, $f \in C^1(R^+, X)$, where $\varepsilon > 0$, $R^+ = [0, \infty)$.

We say that $u : R^+ \rightarrow X$ is a solution of (1.1) if $u \in C^2(R^+, X)$, $u(t) \in D(A)$ (domain of A) for $t \geq 0$ and (1.1) is satisfied on R^+ . Solutions of (1.2) are defined in a similar way. In order to verify the existence of solutions of (1.1) we change it to another more common form. (See [5].) Let

$$u(t; \varepsilon) = e^{-t/2\varepsilon^2} v(t/\varepsilon).$$

Then (1.1) can be replaced by

$$v''(t/\varepsilon) = \left(A + \frac{1}{4\varepsilon^2}\right)v(t/\varepsilon) + \int_0^t K(t-s)e^{(t-s)/2\varepsilon^2} Av(s/\varepsilon)ds + e^{t/2\varepsilon^2} f(t; \varepsilon).$$

Now let $h = t/\varepsilon$, and then change h to t to get

$$(2.1) \quad \begin{aligned} v''(t) &= \left(A + \frac{1}{4\varepsilon^2}\right)v(t) + \int_0^t \hat{K}(t-s)Av(s)ds + \hat{f}(t), \\ v(0; \varepsilon) &= u_0(\varepsilon), \quad v'(0; \varepsilon) = \frac{1}{2\varepsilon}u_0(\varepsilon) + \varepsilon u_1(\varepsilon), \end{aligned}$$

where $(A + \frac{1}{4\varepsilon^2})$ generates a strongly continuous cosine family and

$$\hat{K}(t) = \varepsilon K(\varepsilon t)e^{t/2\varepsilon}, \quad \hat{f}(t) = f(\varepsilon t; \varepsilon)e^{t/2\varepsilon}, \quad t \geq 0.$$

Note that the existence and uniqueness of solutions of (2.1) and (1.2) were obtained in [3, 4, 7, 14, 15] and that we are only interested in singular perturbations in this paper. Thus, we may assume that (1.1) and (1.2) have unique solutions $u(t; \varepsilon)$ and $w(t)$ respectively for every $\varepsilon > 0$.

Now we can state and prove the following result concerning the convergence of solutions, with the following hypotheses:

(H4) $u_0(\varepsilon), w_0 \in D(A)$, $u_0(\varepsilon) \rightarrow w_0$, $\varepsilon^2 u_1(\varepsilon) \rightarrow 0$, as $\varepsilon \rightarrow 0$.

(H5) For any $T > 0$, $f(\cdot; \varepsilon) \rightarrow f(\cdot)$ in $L^1([0, T], X)$ as $\varepsilon \rightarrow 0$.

Theorem 2.1. *Assume that hypotheses (H1)–(H5) are satisfied. Then for any $T > 0$, $u(t; \varepsilon) \rightarrow w(t)$ in X uniformly for $t \in [0, T]$, as $\varepsilon \rightarrow 0$.*

Proof. Define

$$R * H(t) = \int_0^t R(t-s)H(s)ds \quad \text{and} \quad \delta * H = H.$$

Then we can find the solution F of $F + K + F * K = 0$ (see [1, 2, 11, 12]) so that

$$(2.2) \quad (\delta + F) * (\delta + K) = \delta.$$

Now write (1.1) as

$$\varepsilon^2 u''(\varepsilon) + u'(\varepsilon) = (\delta + K) * Au(\varepsilon) + f(\varepsilon).$$

Then we have

$$(\delta + F) * [\varepsilon^2 u''(\varepsilon) + u'(\varepsilon)] = Au(\varepsilon) + (\delta + F) * f(\varepsilon).$$

Hence,

$$\varepsilon^2 u''(\varepsilon) + u'(\varepsilon) = Au(\varepsilon) + (\delta + F) * f(\varepsilon) - F * [\varepsilon^2 u''(\varepsilon) + u'(\varepsilon)].$$

Integration by parts yields

$$\begin{aligned} F * u'(t; \varepsilon) &= \int_0^t F'(t-s)u(s; \varepsilon)ds + F(0)u(t; \varepsilon) - F(t)u_0(\varepsilon), \\ F * u''(t; \varepsilon) &= \int_0^t F''(t-s)u(s; \varepsilon)ds + F(0)u'(t; \varepsilon) - F(t)u_1(\varepsilon) \\ &\quad + F'(0)u(t; \varepsilon) - F'(t)u_0(\varepsilon). \end{aligned}$$

Therefore (1.1) can be replaced by

$$(2.3) \quad \begin{aligned} \varepsilon^2 u''(t; \varepsilon) + u'(t; \varepsilon) &= Au(t; \varepsilon) + \hat{f}(t; \varepsilon), \quad t \geq 0, \\ u(0; \varepsilon) &= u_0(\varepsilon), \quad u'(0; \varepsilon) = u_1(\varepsilon), \end{aligned}$$

with

$$(2.4) \quad \begin{aligned} \hat{f}(t; \varepsilon) &= (\delta + F) * f(t; \varepsilon) - F * [\varepsilon^2 u''(t; \varepsilon) + u'(t; \varepsilon)] \\ &= (\delta + F) * f(t; \varepsilon) - \int_0^t F'(t-s)u(s; \varepsilon) ds - F(0)u(t; \varepsilon) + F(t)u_0(\varepsilon) \\ &\quad - \varepsilon^2 \left[\int_0^t F''(t-s)u(s; \varepsilon) ds + F'(0)u(t; \varepsilon) \right. \\ &\quad \left. - F'(t)u_0(\varepsilon) + F(0)u'(t; \varepsilon) - F(t)u_1(\varepsilon) \right]. \end{aligned}$$

Similarly, (1.2) can be replaced by

$$(2.5) \quad w'(t) = Aw(t) + \hat{f}(t), \quad t \geq 0, \quad w(0) = w_0,$$

with

$$(2.6) \quad \begin{aligned} \hat{f}(t) &= (\delta + F) * f(t) - F * w'(t) \\ &= (\delta + F) * f(t) - \int_0^t F'(t-s)w(s) ds - F(0)w(t) + F(t)w_0. \end{aligned}$$

So we can use results in [5] to get for $t \geq 0$

$$\begin{aligned} w(t) &= S(t)w_0 + \int_0^t S(t-s)\hat{f}(s) ds, \\ u(t; \varepsilon) &= e^{-t/2\varepsilon^2} C(t/\varepsilon)u_0(\varepsilon) + \frac{1}{2}R(t, \varepsilon)u_0(\varepsilon) \\ &\quad + G(t, \varepsilon) \left[\frac{1}{2}u_0(\varepsilon) + \varepsilon^2 u_1(\varepsilon) \right] + \int_0^t G(t-s)\hat{f}(s, \varepsilon) ds, \end{aligned}$$

where $S(\cdot)$, $C(\cdot)$ are given in (H1), $R(\cdot; \varepsilon)$, $G(\cdot; \varepsilon)$ are linear operators defined in [5] using the Bessel functions, and they have the following properties: For some constants α , $\omega > 0$,

$$(P1) \quad \|C(t)\|, \|S(t)\| \leq \alpha e^{\omega^2 t}, \quad t \geq 0, \quad \varepsilon > 0.$$

$$(P2) \quad \|G(t; \varepsilon)\|, \varepsilon^2 \|G'(t; \varepsilon)\| \leq \alpha e^{\omega^2 t}, \quad t \geq 0, \quad \varepsilon > 0.$$

$$(P3) \quad \varepsilon^2 G'(t; \varepsilon) = e^{-t/2\varepsilon^2} C(t/\varepsilon) + \frac{1}{2}[R(t; \varepsilon) - G(t; \varepsilon)].$$

(P4) If $t(\varepsilon) > 0$ for $\varepsilon > 0$ with $t(\varepsilon)/\varepsilon^2 \rightarrow \infty$ as $\varepsilon \rightarrow 0$, then for every $T > 0$,

$$\lim_{\varepsilon \rightarrow 0} \sup_{t(\varepsilon) \leq t \leq T} \|R(t; \varepsilon)x - S(t)x\| = 0 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \sup_{t(\varepsilon) \leq t \leq T} \|G(t; \varepsilon)x - S(t)x\| = 0,$$

uniformly for x in bounded subsets of X .

(P5)

$$\begin{aligned} & \|e^{-t/2\varepsilon^2} C(t/\varepsilon) u_0(\varepsilon) + \tfrac{1}{2} R(t; \varepsilon) u_0(\varepsilon) \\ & \quad + G(t; \varepsilon) [\tfrac{1}{2} u_0(\varepsilon) + \varepsilon^2 u_1(\varepsilon)] - S(t) w_0\| \\ & \leq \alpha e^{\omega^2 t} \left[\varepsilon^2 (1 + \omega^2 t) \|A w_0\| + \|u_0(\varepsilon) - w_0\| + \varepsilon^2 \|u_1(\varepsilon)\| \right], \quad t \geq 0. \end{aligned}$$

Now let $T > 0$ be fixed and consider for $t \in [0, T]$,

$$\begin{aligned} (2.7) \quad u(t; \varepsilon) - w(t) &= e^{-t/2\varepsilon^2} C(t/\varepsilon) u_0(\varepsilon) + \tfrac{1}{2} R(t; \varepsilon) u_0(\varepsilon) \\ & \quad + G(t; \varepsilon) [\tfrac{1}{2} u_0(\varepsilon) + \varepsilon^2 u_1(\varepsilon)] \\ & \quad - S(t) w_0 + \int_0^t \left[G(t-s; \varepsilon) \hat{f}(s; \varepsilon) - S(t-s) \hat{f}(s) \right] ds. \end{aligned}$$

By (H4) and (P5), we can write (2.7) as

$$\begin{aligned} (2.8) \quad u(t; \varepsilon) - w(t) &= 0(\varepsilon, [0, T]) + \int_0^t \left[G(t-s; \varepsilon) \hat{f}(s; \varepsilon) - S(t-s) \hat{f}(s) \right] ds \\ &= 0(\varepsilon, [0, T]) + \int_0^t G(t-s; \varepsilon) \left[\hat{f}(s; \varepsilon) - \hat{f}(s) \right] ds \\ & \quad + \int_0^t \left[G(t-s; \varepsilon) - S(t-s) \right] \hat{f}(s) ds, \end{aligned}$$

where $0(\varepsilon, [0, T])$ satisfies

$$(2.9) \quad 0(\varepsilon, [0, T]) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \quad \text{uniformly for } t \in [0, T].$$

Note that w is locally bounded and $f \in L^1([0, T], X)$; then $\hat{f} \in L^1([0, T], X)$. So from [5],

$$(2.10) \quad \int_0^t \left[G(t-s; \varepsilon) - S(t-s) \right] \hat{f}(s) ds = 0(\varepsilon, [0, T]), \quad t \in [0, T].$$

Next, we have

$$\begin{aligned} (2.11) \quad & \int_0^t G(t-s; \varepsilon) \left[\hat{f}(s; \varepsilon) - \hat{f}(s) \right] ds \\ &= \int_0^t G(t-s; \varepsilon) \left[\hat{f}(s; \varepsilon) - \hat{f}(s) + \varepsilon^2 F(0) u'(s; \varepsilon) \right] ds \\ & \quad - \int_0^t G(t-s; \varepsilon) \varepsilon^2 F(0) u'(s; \varepsilon) ds. \end{aligned}$$

Now from (P3),

$$\begin{aligned}
 & \int_0^t G(t-s; \varepsilon) \varepsilon^2 F(0) u'(s; \varepsilon) ds \\
 &= \varepsilon^2 G(0; \varepsilon) F(0) u(t; \varepsilon) - \varepsilon^2 G(t; \varepsilon) F(0) u_0(\varepsilon) \\
 & \quad + \varepsilon^2 \int_0^t G'(t-s; \varepsilon) F(0) u(s; \varepsilon) ds \\
 &= \varepsilon^2 G(0; \varepsilon) F(0) u(t; \varepsilon) - \varepsilon^2 G(t; \varepsilon) F(0) u_0(\varepsilon) \\
 & \quad + \varepsilon^2 \int_0^t G'(t-s; \varepsilon) F(0) [u(s; \varepsilon) - w(s)] ds \\
 (2.12) \quad & + \varepsilon^2 \int_0^t G'(t-s; \varepsilon) F(0) w(s) ds \\
 &= \varepsilon^2 G(0; \varepsilon) F(0) [u(t; \varepsilon) - w(t)] + \varepsilon^2 G(0; \varepsilon) F(0) w(t) \\
 & \quad - \varepsilon^2 G(t; \varepsilon) F(0) u_0(\varepsilon) \\
 & \quad + \varepsilon^2 \int_0^t G'(t-s; \varepsilon) F(0) [u(s; \varepsilon) - w(s)] ds \\
 & \quad + \int_0^t \left\{ e^{-(t-s)/2\varepsilon^2} C((t-s)/\varepsilon) \right. \\
 & \quad \left. + \frac{1}{2} [R(t-s; \varepsilon) - G(t-s; \varepsilon)] \right\} F(0) w(s) ds.
 \end{aligned}$$

Observe that $w(s)$ is locally bounded, so use property (P4) with $t(\varepsilon) = \varepsilon$ to obtain for any $t, s \in [0, T]$ with $s < t$,

$$(2.13) \quad [R(t-s; \varepsilon) - G(t-s; \varepsilon)] F(0) w(s) \rightarrow 0, \quad \varepsilon \rightarrow 0.$$

Hence, the dominated convergence theorem can be used to prove that

$$(2.14) \quad \int_0^t [R(t-s; \varepsilon) - G(t-s; \varepsilon)] F(0) w(s) ds \rightarrow 0, \quad \varepsilon \rightarrow 0,$$

uniformly for $t \in [0, T]$. Next, assume that $\varepsilon > 0$ is so small that $4\varepsilon\omega^2 \leq 1$. Then from (P1)

$$\begin{aligned}
 & \int_0^t e^{-(t-s)/2\varepsilon^2} \|C((t-s)/\varepsilon)\| ds = \int_0^t e^{-s/2\varepsilon^2} \|C(s/\varepsilon)\| ds \\
 (2.15) \quad & \leq \alpha \int_0^t e^{-s/2\varepsilon^2 + \omega^2 s/\varepsilon} ds = [2\alpha\varepsilon^2/(1-2\varepsilon\omega^2)] [1 - e^{(2\varepsilon\omega^2-1)t/2\varepsilon^2}] \\
 & \leq 4\alpha\varepsilon^2 \rightarrow 0, \quad \varepsilon \rightarrow 0,
 \end{aligned}$$

uniformly for $t \in [0, T]$. Also observe that $w(\cdot)$ is locally bounded and $u_0(\varepsilon)$ has a limit as $\varepsilon \rightarrow 0$. Then from (P2)

$$\varepsilon^2 G(0; \varepsilon) F(0) w(t), \quad \varepsilon^2 G(t; \varepsilon) F(0) u_0(\varepsilon) \longrightarrow 0, \quad \varepsilon \rightarrow 0,$$

uniformly for $t \in [0, T]$, and

$$\begin{aligned} & \left\| \varepsilon^2 \int_0^t G'(t-s; \varepsilon) F(0) [u(s; \varepsilon) - w(s)] ds \right\| \\ & \leq \alpha e^{\omega^2 T} \|F(0)\| \int_0^t \|u(s; \varepsilon) - w(s)\| ds. \end{aligned}$$

Thus by (2.12), (2.14), (2.15), and property (P2) we obtain

$$\begin{aligned} (2.16) \quad & \left\| \int_0^t G(t-s; \varepsilon) \varepsilon^2 F(0) u'(s; \varepsilon) ds - \varepsilon^2 G(0; \varepsilon) F(0) [u(t; \varepsilon) - w(t)] \right\| \\ & \leq (\text{type 1}) + 0(\varepsilon, [0, T]), \end{aligned}$$

where (type 1) is of the form

$$(2.17) \quad (\text{const}) \int_0^t \|u(s; \varepsilon) - w(s)\| ds.$$

Next we have

$$\begin{aligned} & \int_0^t G(t-s; \varepsilon) [\hat{f}(s; \varepsilon) - \hat{f}(s) + \varepsilon^2 F(0) u'(s; \varepsilon)] ds \\ & = \int_0^t G(t-s; \varepsilon) \left\{ [f(s; \varepsilon) - f(s)] + \int_0^s F(s-h) [f(h; \varepsilon) - f(h)] dh \right. \\ & \quad + F(s) [u_0(\varepsilon) - w_0] - \int_0^s F'(s-h) [u(h; \varepsilon) - w(h)] dh \\ & \quad + F(s) \varepsilon^2 u_1(\varepsilon) + \varepsilon^2 F'(s) u_0(\varepsilon) \\ & \quad - [\varepsilon^2 F'(0) + F(0)] [u(s; \varepsilon) - w(s)] \\ & \quad - \varepsilon^2 F'(0) w(s) - \varepsilon^2 \int_0^s F''(s-h) [u(h; \varepsilon) - w(h)] dh \\ & \quad \left. - \varepsilon^2 \int_0^s F''(s-h) w(h) dh \right\} ds. \end{aligned}$$

Note that from (P2)

$$\begin{aligned}
 & \left\| \int_0^t G(t-s; \varepsilon) \int_0^s F(s-h) [f(h; \varepsilon) - f(h)] dh ds \right\| \\
 & \leq \alpha e^{\omega^2 T} \left[\int_0^T \|F(s)\| ds \right] \left[\int_0^T \|f(s; \varepsilon) - f(s)\| ds \right], \\
 & \left\| \int_0^t G(t-s; \varepsilon) F(s) [u_0(\varepsilon) - w_0] ds \right\| \\
 & \leq \alpha e^{\omega^2 T} \|u_0(\varepsilon) - w_0\| \int_0^T \|F(s)\| ds, \\
 & \left\| \int_0^t G(t-s; \varepsilon) \int_0^s F'(s-h) [u(h; \varepsilon) - w(h)] dh ds \right\| \\
 & \leq \alpha e^{\omega^2 T} \left[\int_0^T \|F'(s)\| ds \right] \left[\int_0^T \|u(s; \varepsilon) - w(s)\| ds \right].
 \end{aligned}$$

Other terms can be treated similarly. So it is clear that, with property (P2), hypotheses (H1)–(H5), and the fact that $w(\cdot)$ is locally bounded, we obtain

$$\begin{aligned}
 (2.18) \quad & \left\| \int_0^t G(t-s; \varepsilon) [\hat{f}(s; \varepsilon) - \hat{f}(s) + \varepsilon^2 F(0) u'(s; \varepsilon)] ds \right\| \\
 & \leq (\text{type 1}) + 0(\varepsilon, [0, T]).
 \end{aligned}$$

Combining (2.8), (2.10), (2.11), (2.16), and (2.18), we get

$$(2.19) \quad \|(1 + \varepsilon^2 G(0; \varepsilon) F(0)) [u(t; \varepsilon) - w(t)]\| \leq (\text{type 1}) + 0(\varepsilon, [0, T]).$$

Now assume $\varepsilon > 0$ is so small that $2\|\varepsilon^2 G(0; \varepsilon) F(0)\| < 1$; then

$$\begin{aligned}
 (2.20) \quad & \|u(t; \varepsilon) - w(t)\| \\
 & \leq 0(\varepsilon, [0, T]) + (\text{const}) \int_0^t \|u(s; \varepsilon) - w(s)\| ds, \quad t \in [0, T].
 \end{aligned}$$

So that the Gronwall's inequality [9] can be used to obtain

$$(2.21) \quad \|u(t; \varepsilon) - w(t)\| \leq 0(\varepsilon, [0, T]), \quad t \in [0, T].$$

This proves the theorem. $\square$

Finally, we briefly indicate its applications in viscoelasticity. Let us consider

$$\begin{aligned}
 (2.22) \quad & \rho u_{tt}(t; \rho) + \alpha u_t(t; \rho) = \Delta u(t; \rho) + \int_0^t K(t-s) \Delta u(s; \rho) ds \\
 & + f(t; \rho), \quad t \geq 0, \\
 & u(0; \rho) = u_0(\rho), \quad u_t(0; \rho) = u_1(\rho),
 \end{aligned}$$

in $L^2(\Omega)$, where u is the displacement, ρ is the density per unit area, and α is the coefficient of viscosity of the medium. With appropriate boundary conditions the Laplacian operator Δ in (2.22) generates a strongly continuous cosine family and a strongly continuous semigroup. So with some convergence

conditions on initial data and $f(t; \varepsilon)$ and smoothness conditions on $K(\cdot)$, Theorems 2.1 can be used to show that when density $\rho \rightarrow 0$, solutions of (2.22) will converge to solutions of the "limiting" heat equation

$$(2.23) \quad \alpha w_t(t) = \Delta w(t) + \int_0^t K(t-s) \Delta w(s) ds + f(t), \quad t \geq 0, \\ w(0) = w_0.$$

Details are omitted here. This result also relates to a concept called "change the type" (from hyperbolic to parabolic).

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