ON THE JACOBIAN OF THE KLEIN CURVE

DESPINA T. PRAPAVESSI

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ABSTRACT. It is known that the Jacobian J of the Klein curve is isogenous to E^3 for a certain elliptic curve E . We compute explicit equations for E and prove that J is in fact isomorphic to E^3 . We also identify the subgroup of J generated by the image of the Weierstrass points of the curve under an Albanese embedding, and we show that it is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/7\mathbb{Z})^3$.

1. INTRODUCTION

Let ζ be a primitive 7th root of 1 in $\overline{\mathbb{Q}}$, and algebraic closure of $\mathbb{Q}$. Let $k = \mathbb{Q}(\zeta)$, and $\mu_i = \zeta^i + \zeta^{-i}$. The canonical embedding of the cyclic Fermat quotient curve $t^7 = s(1-s)^2$ in $\mathbb{P}_{\mathbb{Q}}^2$ is the Klein curve X , given by the equation

$$x^3y + y^3z + z^3x = 0.$$

The curve X has genus $g = 3$ and it is well known that its group of automorphisms is isomorphic to $\mathrm{PSL}_2(\mathbb{F}_7)$ which has order 168, the maximum for a curve X of genus 3. Generators of the group are [2, 5]

$$\sigma(x, y, z) = (y, z, x) \quad \text{of order 3,}$$

$$\tau(x, y, z) = \left(\mu_1 x + \frac{1}{\mu_3} y + z, \frac{1}{\mu_3} x + y + \mu_1 z, x + \mu_1 y + \frac{1}{\mu_3} z \right) \quad \text{of order 2,}$$

and

$$\varepsilon(x, y, z) = (\zeta x, \zeta^5 y, z) \quad \text{of order 7.}$$

We will explicitly compute a subgroup of the Jacobian of X over K isomorphic to $(\mathbb{Z}/7\mathbb{Z})^3$. The existence of this group was predicted by Greenberg in [3].

We let $\infty_1, \infty_2, \infty_3$ denote the three cusps of X , given by

$$\infty_1 = (1, 0, 0), \quad \infty_2 = (0, 1, 0), \quad \infty_3 = (0, 0, 1).$$

We let J_X denote the Jacobian of X . To simplify the notation, we also denote by σ, τ , and ε the induced endomorphisms on J_X . Note that by mapping ζ to ε we obtain on J_X complex multiplication by $\mathbb{Z}[\zeta]$. We will identify $\mathbb{Z}[\zeta]$ with its image. We let

$$\pi = \zeta - 1 \in \mathrm{End}(J_X).$$

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The divisors $\infty_2 - \infty_1$ and $\infty_3 - \infty_1$ are both fixed by ζ , and hence have exact order π . (They are nonprincipal, since the genus of X is nonzero.)

Recall that P is a Weierstrass point on a curve of genus g , if there exists a nonconstant function on the curve with a pole at P of order at most g , and no other poles [9].

Let W denote the set of Weierstrass points on X . We have that

$$(1.1) \quad W = \{\theta(\infty_1) \mid \theta \in \text{Aut}(X)\}.$$

To show this notice that W is stable under the action of $\text{Aut}(X)$. Clearly $\infty_1 \in W$ since the divisor of $\frac{x}{y}$ is $\infty_2 + 2\infty_3 - 3\infty_1$. A direct computation shows that the set on the right side of (1.1) is $\{(\zeta^i \mu_j, \zeta^{5i} \frac{1}{\mu_{3j}}, 1) \mid i = 0, 1, \dots, 6, j = 1, 2, 3\} \cup \{\infty_1, \infty_2, \infty_3\}$. In particular it has order 24, which is the maximum number of Weierstrass points on a curve of genus 3.

Let

$$P = \tau\infty_1 = \left(\mu_1, \frac{1}{\mu_3}, 1\right), \quad D = P + \sigma P - 2\infty_1.$$

We will prove:

Theorem 1. *The subgroup $J_{X,W}$ of $J_X(K)$, generated by the divisor classes of divisors of degree 0 supported on W , is the kernel of $2\pi^3$. In particular the class of D generates the kernel of π^3 as a $\mathbb{Z}[\zeta]$ module.*

Notice that W is the cuspidal torsion packet [1] of X , studied by Coleman in [2].

Remark. Using similar methods to the ones used here, we have also computed explicitly a subgroup of the Jacobian of the cyclic Fermat quotient curve $t^{13} = s(1-s)^3$ over the 13th cyclotomic field, isomorphic to $\mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/13\mathbb{Z})^3$.

We also prove here:

Theorem 2. *J_X is isomorphic over K to the product $E_0 \times E_1 \times E_2$ of three isomorphic elliptic curves with Weierstrass equations given by*

$$(1.2) \quad E_i: y^2 + 3\zeta^{4i}xy + \zeta^{5i}y = x^3 - 2\zeta^{2i}x - 3\zeta^{3i}, \quad i = 0, 1, 2.$$

Remark. The proof of Theorem 2 can be extended to prove that the Jacobian of the curve $t^p = s(1-s)^r$, with p prime, $p \equiv 1 \pmod{6}$, and $r^2 + r + 1 = 0 \pmod{p}$ is isomorphic over the p th cyclotomic field to the product of three isomorphic simple factors. Koblitz and Rohrlich have shown in [6] that the Jacobian of these curves is isogenous to such a product.

In §2 we compute the equations for the elliptic curves in (1.2) which are the quotient curves of X under the action of $\varepsilon^i \sigma \varepsilon^{-i}$ for $i = 0, 1$, or 2 . We also compute equations for an isogeny ϕ_* from J_X to the product of these three curves. In §3 we study the action of ϕ_* on $J_X[\pi^3]$. Finally in §4 we prove Theorems 1 and 2.

2. THE ISOGENY ϕ_* OF J_X TO THE PRODUCT $E_0 \times E_1 \times E_2$

Let $\sigma_i = \varepsilon^i \sigma \varepsilon^{-i} \in \text{Aut}(X)$. In particular $\sigma_0 = \sigma$. Since σ_i has order 3 it generates a cyclic subgroup $\langle \sigma_i \rangle$ of $\text{Aut}_K(K(X))$ of order 3. Let F_i denote the fixed field of $\langle \sigma_i \rangle$. Then $K(X)$ is a finite separable extension of F_i of degree 3 and the inclusion of fields $F_i \subseteq K(X)$ corresponds to a morphism of curves

$$\phi_i: X \rightarrow E_i$$

of degree 3. By Hurwitz's theorem E_i has genus 1. Then ϕ_i induces a unique homomorphism of abelian varieties

$$(\phi_i)_*: J_X \rightarrow E_i,$$

We have:

Proposition 3. *The morphism of abelian varieties $\phi_* := \prod_{i=0}^2 (\phi_i)_*$,*

$$\phi_*: J_X \rightarrow E_0 \times E_1 \times E_2,$$

is an isogeny.

Proof of Proposition 3. Observe that both J_X and $E_0 \times E_1 \times E_2$ have dimension 3. Consider the induced homomorphism of the tangent spaces

$$d(\phi_*): T_{J_X,0} \rightarrow T_{E_0 \times E_1 \times E_2,0},$$

where $T_{A,0}$ denotes the tangent space to the abelian variety A , at 0. It suffices to show that $d\phi_*$ is an isomorphism.

Identifying $T_{J_X,0}$ with the vector space of holomorphic differentials on X , we compute a basis $\{\omega_1, \omega_2, \omega_3\}$ for $T_{J_X,0}$ from the known basis of differentials on the Fermat curve computed in [7, I.7]. We find that

$$\omega_1 = \frac{1}{7} \frac{y^4}{x^3} d\left(\frac{x^3}{y^2}\right), \quad \omega_2 = \frac{1}{7} \frac{y^3}{x^2} d\left(\frac{x^3}{y^2}\right), \quad \omega_3 = \frac{1}{7} \frac{y^3}{x^3} d\left(\frac{x^3}{y^2}\right).$$

Next, notice that $T_{E_0 \times E_1 \times E_2,0}$ is a direct sum of the $T_{E_i,0}$'s for $i = 0, 1$, or 2 , and a basis for $T_{E_i,0}$ is given by the image under $d\phi_i$ of any nonzero holomorphic differential which is invariant under the action of $\langle \sigma_i \rangle$. In particular $d\phi_i(-\omega_1 + \zeta^{4i}\omega_2 - \zeta^{5i}\omega_3)$ is a basis for $T_{E_i,0}$. Hence we obtain

$$d\phi_* = \begin{bmatrix} -1 & -1 & -1 \\ 1 & \zeta^4 & \zeta \\ -1 & -\zeta^5 & -\zeta^3 \end{bmatrix}^{-1},$$

and since the determinant of the above matrix is nonzero, Proposition 3 follows.

Next we will show:

Proposition 4. *With notation as above, E_i is the elliptic curve with zero element the point $O = (0, 1, 0)$, and Weierstrass equation*

$$y^2 + 3\zeta^{4i}xy + \zeta^{5i}y = x^3 - 2\zeta^{2i}x - 3\zeta^{3i}, \quad i = 0, 1, 2.$$

The finite morphism $\phi_i: X \rightarrow E_i$ is given by

$$(2.1) \quad \phi_i(x, y) = (-w_i, v_i),$$

where

$$w_i = x + \zeta^{6i} \frac{1}{y} + \zeta^{4i} \frac{y}{x}, \quad v_i = y + \zeta^{6i} \frac{1}{x} + \zeta^{2i} \frac{x}{y}.$$

Furthermore each E_i has CM by $\mathbf{Z}[(1 + \sqrt{-7})/2]$, and for $i \neq j$, E_i is canonically isomorphic to E_j .

Proof of Proposition 4. Notice that $w_i = \text{Tr}_{F_i}^{K(X)}(x)$, and $v_i = \text{Tr}_{F_i}^{K(X)}(y)$, where Tr denotes the trace map. Since w_i and v_i satisfy

$$v_i^2 - 3\zeta^{4i}v_iw_i + \zeta^{5i}v_i = -w_i^3 + 2\zeta^{2i}w_i - 3\zeta^{3i}$$

they generate over K a subfield of $K(\mathbf{X})$ which is contained in F_i . Since the degree of the extension $K(\mathbf{X})/K(w_i, v_i)$ is 3, we have $K(w_i, v_i) = F_i$.

Notice that the morphism ϕ_i is only ramified at the points $(\zeta^i \omega, \zeta^{5i} \omega^2)$ and $(\zeta^i \omega^2, \zeta^{5i} \omega)$, where ω is a cubic root of 1 in $\overline{\mathbf{Q}}$. The fibre over $\phi_i(P)$ is $\{P, \sigma_i(P), \sigma_i^2(P)\}$. Also notice that $\mathbf{E}_i$ has discriminant $-\zeta^{6i} \cdot 7^3$, and j -invariant $-5^3 \cdot 3^3$.

For $i \neq j$, $K(\mathbf{E}_i)$ and $K(\mathbf{E}_j)$ are intermediate extensions of $K(\mathbf{X})/K(\mathbf{X})^{(\sigma, \varepsilon)}$ (where $K(\mathbf{X})^{(\sigma, \varepsilon)}$ is the fixed field of $\langle \sigma, \varepsilon \rangle$), corresponding to the conjugate subgroups $\langle \sigma_i \rangle$ and $\langle \sigma_j \rangle$. Therefore the function fields and hence the curves are isomorphic.

It remains to show that $\mathbf{E}_i$ has CM by the maximal order in $\mathbf{Q}(\sqrt{-7})$. Since the $\mathbf{E}_i$'s are isomorphic it suffices to work with $i = 0$. Define $e \in \text{End}(\mathbf{E}_0)$ by

$$e(P) = \phi_0(\zeta \overline{P}) + \phi_0(\zeta^2 \overline{P}) + \phi_0(\zeta^4 \overline{P}),$$

where $\overline{P} \in \mathbf{X}$ is such that $\phi_0(\overline{P}) = P$. Existence of $\overline{P}$ follows by surjectivity of ϕ_0 , and one can easily check that e is well defined, since $\zeta \sigma = \sigma \zeta^2$ in $\text{Aut}(\mathbf{X})$. Furthermore we have

$$(e^2 + e + 2)(P) = 2 \sum_{i=1}^7 \phi_0(\zeta^i \overline{P}) = 0.$$

Hence by mapping the quotient $(-1 + \sqrt{-7})/2$ to e we obtain an endomorphism of $\mathbf{Z}[(1 + \sqrt{-7})/2]$ to $\text{End}(\mathbf{E}_0)$, which must be an isomorphism since $\mathbf{Z}[(1 + \sqrt{-7})/2]$ is the maximal order in $\mathbf{Q}(\sqrt{-7})$. This completes the proof of Proposition 4.

The following lemma will be used in §3.

Lemma 5. *With notation as above, on $\mathbf{J}_\mathbf{X}$ we have*

$$\text{Ker}(\phi_i)_* = \text{Ker}(1 + \sigma_i + \sigma_i^2).$$

Furthermore, for $i \neq j$, $\text{Ker}(\phi_i)_* \neq \text{Ker}(\phi_j)_*$.

Proof of Lemma 5. If $D \in \mathbf{J}_\mathbf{X}$ is such that $(\phi_i)_* D = (f)$ for some $f \in K(\mathbf{E}_i)$, then $(1 + \sigma_i + \sigma_i^2)D = (f \circ \phi_i)$; hence, on $\mathbf{J}_\mathbf{X}$, $\text{Ker}(\phi_i)_* \subseteq \text{Ker}(1 + \sigma_i + \sigma_i^2)$. Conversely, if $(1 + \sigma_i + \sigma_i^2)D = (g)$, for some $g \in K(\mathbf{X})$, then $(\sigma_i g/g) = 0$; hence, $\sigma_i g = \lambda g$, $\lambda \in K$. Hence

$$(\sigma_i + \sigma_i^2 + 1)g = \lambda^3(1 + \sigma_i + \sigma_i^2)g.$$

Since K contains no nontrivial cubic roots of 1, $\lambda = 1$, that is $\sigma_i(g) = g$; therefore $g \in F_i$. Hence $g = f \circ \phi_i$, with $f \in K(\mathbf{E}_i)$. Therefore $(\phi_i)_* D = (f)$, and hence $\text{Ker}(1 + \sigma_i + \sigma_i^2) \subseteq \text{Ker}(\phi_i)_*$. To prove that for $i \neq j$, $\text{Ker}(\phi_i)_* \neq \text{Ker}(\phi_j)_*$, we compute $\mathbf{E}_0[2]$, the 2-torsion subgroup of $\mathbf{E}_0$. We have

$$(2.2) \quad \mathbf{E}_0[2] = \{R, \overline{R}, R + \overline{R}, \mathbf{O}\},$$

where $R = ((-13 + \sqrt{-7})/8, (31 + 3\sqrt{-7})/16)$. (Observe that $R + \overline{R} = (1, -2) \in \mathbf{E}_0(\mathbf{Q})$.) Since $\mathbf{D} = P + \sigma P - 2\mathbf{O}$, a direct computation gives $\phi_0(P) = (1, -2)$; hence

$$(2.3) \quad (\phi_0)_* \mathbf{D} = 2(R + \overline{R}) - 2\mathbf{O} = \mathbf{O},$$

while for $i = 1$ or 2 ,

$$(2.4) \quad (\phi_i)_* \mathbf{D} = (\zeta^i(\mu_1 - 1), \zeta^{5i}(1 + \mu_1(\zeta + \zeta^2 + \zeta^4)^2)) \neq \mathbf{O}.$$

(In fact one can show that $(\phi_i)_* \mathbf{D} \in \mathbf{E}_i[\sqrt{-7}]$ for $i \neq 0$, but we will not use this here.) Using the isomorphism of the curves $\mathbf{E}_i$, it follows that the kernels of the maps $(\phi_i)_*$ are pairwise distinct. This completes the proof of Lemma 5.

3. THE ACTION OF ϕ_* ON $\mathbf{J}_X[\pi^3]$

Let $R = \mathbf{Z}[\zeta]/(\pi^3)$. We fix a generator β of $\mathbf{J}_X[\pi^3]$ as a principal R module. Then β has exact order π^3 and the action of σ on $\mathbf{J}_X[\pi^3]$ is completely determined by its action on β . Since $\sigma \in \text{Aut}(\mathbf{J}_X)$, we have

$$\sigma(\beta) = \alpha_\sigma \beta, \quad \alpha_\sigma \in R^*.$$

We will now determine α_σ . Notice that R^* is a $\langle \sigma \rangle$ module via the action

$$(3.1) \quad \left(\sum_{i=1}^7 a_i \zeta^i \right)^\sigma = a_i \zeta^{4i}, \quad a_i \in \mathbf{Z},$$

and since $\sigma\zeta = \zeta^4\sigma$ on $\mathbf{J}_X$, the map $\sigma \rightarrow \alpha_\sigma$ gives rise to a cocycle in $H^1(\langle \sigma \rangle, R^*)$. Since $\sigma^3 = 1$, we have $\beta = \sigma^3\beta = \alpha_\sigma^{1+\sigma+\sigma^2}\beta$, and therefore

$$(3.2) \quad \alpha_\sigma^{1+\sigma+\sigma^2} = 1$$

in R . We will prove:

Lemma 6. *With notation as above, $\alpha_\sigma = 2$.*

For the proof of Lemma 6 we will need Lemma 7 stated below. Consider the exact sequence of $\langle \sigma \rangle$ modules:

$$0 \rightarrow (1 + \pi\mathbf{Z}[\zeta]/(\pi^3)) \rightarrow R^* \rightarrow (\mathbf{Z}/7\mathbf{Z})^* \rightarrow 0$$

with the action of σ as described in (3.1). We get the long exact sequence of cohomology:

$$\begin{aligned} \cdots \rightarrow H^1(\langle \sigma \rangle, 1 + \pi\mathbf{Z}[\zeta]/(\pi^3)) &\rightarrow H^1(\langle \sigma \rangle, R^*) \\ &\rightarrow H^1(\langle \sigma \rangle, (\mathbf{Z}/7\mathbf{Z})^*) \rightarrow H^2(\langle \sigma \rangle, 1 + \pi\mathbf{Z}[\zeta]/(\pi^3)) \rightarrow \cdots \end{aligned}$$

We will prove:

Lemma 7. *Let $B = 1 + \pi\mathbf{Z}[\zeta]/(\pi^3)$. Then for $i = 1$ and 2 we have $H^i(\langle \sigma \rangle, B) = 0$.*

Proof of Lemma 7. Since $\langle \sigma \rangle$ is finite and cyclic, we have [8, Theorem 10.35]

$$H^1(\langle \sigma \rangle, B) = B_0/B^{1-\sigma},$$

while

$$H^2(\langle \sigma \rangle, B) = B_1/B^{(1+\sigma+\sigma^2)},$$

where $B_0 = \{b \in B \mid b^{1+\sigma+\sigma^2} = 1\}$, and $B_1 = \{b \in B \mid b^\sigma = b\}$. Also it follows from (3.1) that in R we have

$$(3.3) \quad \pi^\sigma \equiv -\pi^2 + 4\pi \pmod{\pi^3}, \quad \pi^{\sigma^2} \equiv \pi^2 + 2\pi \pmod{\pi^3}.$$

A quick computation yields that $B_0 = B^{1-\sigma} = B$, and that $B_1 = B^{1+\sigma+\sigma^2} = \{1\}$. So Lemma 7 follows.

Proof of Lemma 6. It follows from Lemma 7 that the homomorphism $H^1(\langle\sigma\rangle, R^*) \rightarrow H^1(\langle\sigma\rangle, (\mathbf{Z}/7\mathbf{Z})^*)$ is an isomorphism. Hence $\alpha_\sigma \in (\mathbf{Z}/7\mathbf{Z})^*$ and (3.2) gives the result that $\alpha_\sigma^3 = 1$, hence $\alpha_\sigma = 1, 2$, or 4 . We will next rule out the cases $\alpha_\sigma = 1$ and $\alpha_\sigma = 4$.

We have

$$\mathbf{J}_X[\pi^3] = \{(a + b\pi + c\pi^2)\beta \mid a, b, c \in (\mathbf{Z}/7\mathbf{Z})\}.$$

Using Lemma 5 and (3.3) we can compute $\text{Ker}(\phi_i)_*$ for $i = 0, 1$, or 2 .

If $\alpha_\sigma = 1$, we obtain

$$(1 + \sigma + \sigma_i^2)(a + b\pi + c\pi^2)\beta = 3a\beta \quad \text{for } i = 0, 1, 2.$$

But this implies that for $i = 0, 1$, or 2 , we have

$$\text{Ker}(\phi_i)_* = \{(b\pi + c\pi^2)\beta \mid a, c \in (\mathbf{Z}/7\mathbf{Z})\} = \mathbf{J}_X[\pi^2],$$

which contradicts Lemma 5.

If $\alpha_\sigma = 4$, we obtain

$$(1 + \sigma + \sigma^2)(a + b\pi + c\pi^2)\beta = (3c - 2b)\pi^2\beta;$$

hence $\mathbf{J}_X[\pi] = \{(c\pi^2)\beta \mid c \in (\mathbf{Z}/7\mathbf{Z})\}$ is not contained in $\text{Ker}(\phi_0)_*$. This is a contradiction, because $(\phi_0)_*(\infty_2 - \infty_1) = \mathbf{O}$, and as noted in the introduction, $\infty_2 - \infty_1$ generates the π -torsion on $\mathbf{J}_X$. Hence $\alpha_\sigma = 2$.

Next we will prove:

Lemma 8. *With notation as above, for $i = 0, 1, 2$ we have $\text{Ker}(\phi_i)_* \cap \mathbf{J}_X[\pi^2] = \mathbf{J}_X[\pi]$.*

Proof of Lemma 8. Using Lemmas 5 and 6 along with (3.3), a direct computation gives

$$\text{Ker}(\phi_i)_* \cap \mathbf{J}_X[\pi^3] = \begin{cases} \{(a + c\pi^2)\beta \mid a, c \in (\mathbf{Z}/7\mathbf{Z})\} & \text{if } i = 0, \\ \{(a - a\pi + c\pi^2)\beta \mid a, c \in (\mathbf{Z}/7\mathbf{Z})\} & \text{if } i = 1, \\ \{(a + 2a\pi + c\pi^2)\beta \mid a, c \in (\mathbf{Z}/7\mathbf{Z})\} & \text{if } i = 2. \end{cases}$$

Then Lemma 8 follows.

Next we will show:

Lemma 9. *With notation as above we have*

$$\text{Ker}(\phi_0)_* \cap \text{Ker}(\phi_1)_* \cap \text{Ker}(\phi_2)_* = \mathbf{J}_X[\pi].$$

Proof of Lemma 9. Suppose that for $i = 0, 1$, or 2 , $D \in \text{Ker}(\phi_i)_*$. Then by Lemma 5 $(1 + \sigma_i + \sigma_i^2)D = 0$. Using the relations

$$1 + \sigma_i + \sigma_i^2 = 1 + \zeta^{4i}\sigma + \zeta^{2i}\sigma^2, \quad i = 1, 2,$$

it is easy to check that in $\text{End}(\mathbf{J}_X)$ we have

$$(\zeta - 1)^2(\zeta + 1)\zeta(\zeta^2 + \zeta + 1)\sigma = \zeta^6(1 + \sigma + \sigma^2) - (1 + \zeta^6)(1 + \sigma_1 + \sigma_1^2) + (1 + \sigma_2 + \sigma_2^2).$$

It follows that $(\zeta - 1)^2D = 0$, since $\zeta + 1$, ζ , $\zeta^2 + \zeta + 1$, and σ are all units in $\text{End}(\mathbf{J}_X)$. Hence $D \in (\mathbf{J}_X[\pi^2])$. Then by Lemma 8 we have that $D \in \mathbf{J}_X[\pi]$. The other inclusion is obvious.

4. THE PROOF OF THEOREMS 1 AND 2

Recall that $\mathbf{D} = P + \sigma P - 2\infty_1$.

Proof of Lemma 1. We will first show that $\mathbf{D}$ has exact order π^3 . Using Lemmas 7 and 8, it suffices to show that

- (1) $\mathbf{D} \in \mathbf{J}_X[\pi^3]$,
- (2) $\mathbf{D} \in \text{Ker}(\phi_0)_*$, and
- (3) $\mathbf{D} \notin \text{Ker}(\phi_1)_*$.

To prove (1), consider $f \in K(X)$ given by

$$f = \zeta \mu_2 x^2 + y^2 + \zeta^3 xy + \frac{\zeta^4}{\mu_3} y + \frac{\mu_2}{\zeta} x + \frac{\zeta^5}{\mu_3}.$$

Then the divisor of f is

$$(f) = ((\zeta + \zeta^2 + \zeta^4)(P + \sigma P)) + P + \sigma^2 P - 6\infty_2 - 2\infty_1.$$

Since P , σP , and $\sigma^2 P$ are fixed under complex conjugation we have

$$\left(\frac{f}{\bar{f}}\right) = ((\zeta + \zeta^2 + \zeta^4) - (\zeta^3 + \zeta^5 + \zeta^6))\mathbf{D},$$

where $\bar{f}$ denotes the complex conjugate of f . That is, we have $(\pi^3)\mathbf{D} = 0$, because

$$(\zeta + \zeta^2 + \zeta^4) - (\zeta^3 + \zeta^5 + \zeta^6) = (\zeta - 1)^3(\zeta + 1)^2(\zeta^2 + 1),$$

and $\zeta + 1$, $\zeta^2 + 1$ are units in $\mathbf{Z}[\zeta]$.

Notice also that (2) and (3) follow from (2.3) and (2.4).

To complete the proof of Theorem 1 it remains to show that $\mathbf{J}_{X, \mathbf{w}} = \mathbf{J}_X[2\pi^3]$. Notice that

$$\mathbf{D}_0 = P + \sigma P + \sigma^2 P - \infty_1 - \infty_2 - \infty_3 \in \mathbf{J}_X(\mathbf{Q})$$

has exact order 2 since $\mathbf{D}_0 = (\phi_0)_*^{-1}((2, -1) - \mathbf{O})$. But then the points $\mathbf{D}_i = \zeta^i \mathbf{D}_0$, $i = 0, 1, \dots, 5$ are independent [4, Lemma 1.3] and hence

$$(\mathbf{Z}/2\mathbf{Z})^6 \subseteq \mathbf{J}_{X, \mathbf{w}}.$$

Therefore $\mathbf{J}_X[2\pi^3] \subseteq \mathbf{J}_{X, \mathbf{w}}$. Conversely, since $P - \infty_1 = \mathbf{D}_0 - \mathbf{D} + (\infty_2 + \infty_3 - 2\infty_1)$, it follows that $(P - \infty_1) \in \mathbf{J}_X[2\pi^3]$. Therefore $\mathbf{J}_{X, \mathbf{w}} \subseteq \mathbf{J}_X[2\pi^3]$. This completes the proof of Theorem 1.

Proof of Theorem 2. Consider the isogeny

$$\phi: \mathbf{J}_X \rightarrow \mathbf{E}_0 \times \mathbf{E}_1 \times \mathbf{E}_2$$

studied in §2. We have from Lemma 9 that $\text{Ker}(\phi)_* = \mathbf{J}_X[\pi]$. Hence we may define an isomorphism ι_ϕ as follows: Given $D \in \mathbf{J}_X$, let $\bar{D}$ be such that $\pi\bar{D} = D$. Since π is a surjective endomorphism, such a $\bar{D}$ always exists. Then define $\iota_\phi(D) = \phi_*(\bar{D})$. It is easy to check that ι_ϕ is well defined and a birational isomorphism, hence an isomorphism of abelian varieties. This completes the proof of Theorem 2.

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DEPARTMENT OF MATHEMATICS, DIABLO VALLEY COLLEGE, PLEASANT HILL, CALIFORNIA 94523