ON THE MODULAR EQUATIONS OF DEGREE 3

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ABSTRACT. In this paper, we use the twelve Jacobian elliptic functions to derive a collection of 24 identities which are essential in the study of the modular equations of degree 3.

1. INTRODUCTION

One of the most interesting consequences of the Ramanujan's ${}_1\psi_1$ summation is the following well-known identity:

$$(1.1) \quad \sum_{n=-\infty}^{\infty} \frac{\alpha^n}{1-tq^{2n}} = \frac{i}{2} \vartheta'_1 \frac{\vartheta_1(x+y, q)}{\vartheta_1(x, q)\vartheta_1(y, q)}$$

where $\alpha = e^{2ix}$, $t = e^{2iy}$, and $|q| < 1$. This identity is valid for $|q^2| < |\alpha| < 1$ and $|t| \neq 1$. When there is no confusion, we will use $\vartheta_i(z)$ to denote $\vartheta_i(z, q)$, ϑ_i to denote $\vartheta_i(0, q)$ for $i = 1, 2, 3, 4$, and ϑ'_1 to denote $\vartheta'_1(0, q)$. We assume the reader is familiar with the basic notation and properties of the theta functions mentioned in [7, Chapters 21 and 22].

Perhaps the most important feature of this identity is that it contains the Fourier series expansions of all the twelve Jacobian elliptic functions. For example, to deduce the Fourier series expansion for dnz , we recall that by definition

$$dnz = \frac{\vartheta_4 \vartheta_3(z/\vartheta_3^2)}{\vartheta_3 \vartheta_4(z/\vartheta_3^2)}$$

or equivalently

$$dn\left(\frac{2Kz}{\pi}\right) = \frac{\vartheta_4 \vartheta_3(z)}{\vartheta_3 \vartheta_4(z)}, \quad \text{where } K = \frac{\pi}{2} \vartheta_3^2.$$

Now in (1.1), choosing $y = \frac{\pi}{2}$ and $x = z + \frac{\pi\tau}{2}$ and using the facts that $\vartheta_1(\frac{\pi}{2}) = \vartheta_2$, $\vartheta'_1 = \vartheta_2\vartheta_3\vartheta_4$ and

$$\begin{aligned} \vartheta_1\left(z + \frac{\pi}{2} + \frac{\pi\tau}{2}\right) &= q^{-1/4} e^{-iz} \vartheta_3(z), \\ \vartheta_1\left(z + \frac{\pi\tau}{2}\right) &= iq^{-1/4} e^{-iz} \vartheta_4(z), \end{aligned}$$

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we see that (1.1) becomes

$$\begin{aligned}
 \vartheta_3 \vartheta_4 \frac{\vartheta_3(z)}{\vartheta_4(z)} &= 2 \sum_{n=-\infty}^{\infty} \frac{q^n e^{2inz}}{1+q^{2n}} = 1 + 4 \sum_{n=1}^{\infty} \frac{q^n \cos 2nz}{1+q^{2n}} \\
 (1.2) \qquad &= 1 + 2 \sum_{n=1}^{\infty} \frac{q^n (t^n + t^{-n})}{1+q^{2n}} \quad (t = e^{2iz}).
 \end{aligned}$$

Thus, the Fourier series expansion of $dn(\frac{2Kz}{\pi})$ is

$$dn\left(\frac{2Kz}{\pi}\right) = \frac{\pi}{2K} + \frac{2\pi}{K} \sum_{n=1}^{\infty} \frac{q^n \cos 2nz}{1+q^{2n}}.$$

It is worthwhile to point out that if we choose $z = 0$ in (1.2), we obtain the following famous Lambert series for ϑ_3^2 :

$$(1.3) \qquad \vartheta_3^2 = 1 + 4 \sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}}.$$

Since the Jacobian elliptic functions are essentially the constant multiples of $\vartheta_i(z)/\vartheta_j(z)$, where $i \neq j$ and $i, j = 1, 2, 3, 4$, we see that there are twelve of them. And by proper replacements of x and y in (1.1), we can easily obtain the Fourier series expansions of these twelve elliptic functions. The complete listing of the twelve Jacobian elliptic functions is given in [7, pp. 511–512].

The main purpose of this note is to provide a unified treatment in obtaining a collection of well-known Lambert series which are essential for the study of the modular equations of degree 3.

In the following section, we will use (1.2) to illustrate our general method and will list the 24 identities obtained by this method in §3. In §4 we will mention some immediate consequences of these 24 identities. Concluding remarks are given in §5.

2. THE METHOD

We now use the identity (1.2) to describe our method. First we should state that throughout the remainder of this paper, we will use $\vartheta_i(q^\alpha)$ to denote $\vartheta_i(0, q^\alpha)$, $i = 2, 3, 4$.

We first replace q by $q^{3/2}$ and choose $z = \frac{\pi\tau}{4}$. Then (1.2) becomes

$$\begin{aligned}
 \vartheta_3(q^{3/2})\vartheta_4(q^{3/2})\frac{\vartheta_3(\frac{\pi\tau}{4}, q^{3/2})}{\vartheta_4(\frac{\pi\tau}{4}, q^{3/2})} &= 1 + 2 \sum_{n=1}^{\infty} \frac{q^{3n/2}(q^{n/2} + q^{-n/2})}{1+q^{3n}} \\
 (2.1) \qquad &= 1 + 2 \sum_{n=1}^{\infty} \frac{q^n}{1+q^{3n}} + \frac{q^{2n}}{1+q^{3n}} \\
 &= 1 + 2 \sum_{n=1}^{\infty} (-1)^n \left(\frac{q^{3n+1}}{1-q^{3n+1}} + \frac{q^{3n+2}}{1-q^{3n+2}} \right).
 \end{aligned}$$

On the other hand, since [7, p. 469]

$$\begin{aligned}\vartheta_3(z, q) &= \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n-1}e^{2iz})(1 + q^{2n-1}e^{-2iz}), \\ \vartheta_4(z, q) &= \prod_{n=1}^{\infty} (1 - q^{2n})(1 - q^{2n-1}e^{2iz})(1 - q^{2n-1}e^{-2iz}),\end{aligned}$$

we see that the left-hand side of (2.1) is

$$\begin{aligned}\prod_{n=1}^{\infty} (1 - q^{3n})^2 (1 - q^{6n-3})^2 \frac{(1 + q^{3n-2})(1 + q^{3n-1})}{(1 - q^{3n-2})(1 - q^{3n-1})} \\ = \prod_{n=1}^{\infty} \frac{(1 - q^{3n})^3 (1 + q^n)}{(1 + q^{3n})^3 (1 - q^n)} = \frac{\vartheta_4^3(q^3)}{\vartheta_4(q)}.\end{aligned}$$

Therefore, we obtain

$$(2.2) \quad \frac{\vartheta_4^3(q^3)}{\vartheta_4(q)} = 1 + 2 \sum_{n=0}^{\infty} (-1)^n \left(\frac{q^{3n+1}}{1 - q^{3n+1}} + \frac{q^{3n+2}}{1 - q^{3n+2}} \right).$$

Now replacing q by $-q$, the identity (2.2) becomes

$$(2.3) \quad \frac{\vartheta_3^3(q^3)}{\vartheta_3(q)} = 1 - 2 \sum_{n=1}^{\infty} \frac{q^{6n+1}}{1 + q^{6n+1}} - \frac{q^{6n+2}}{1 - q^{6n+2}} + \frac{q^{6n+4}}{1 - q^{6n+4}} - \frac{q^{6n+5}}{1 + q^{6n+5}}.$$

It is natural to ask whether more identities of this type can be obtained by choosing q and z properly. This is indeed the case. In fact, four more identities of this type will result from this process. To see this, we again replace q in (1.2) by $q^{3/2}$ and now choose $z = \frac{\pi\tau}{4} + \frac{\pi}{2}$; then (1.2) yields

$$\begin{aligned}(2.4) \quad \vartheta_4(q)\vartheta_4(q^3) &= 1 + 2 \sum_{n=1}^{\infty} (-1)^n \left(\frac{q^n + q^{2n}}{1 + q^{3n}} \right) \\ &= 1 - 2 \sum_{n=0}^{\infty} (-1)^n \left(\frac{q^{3n+1}}{1 + q^{3n+1}} + \frac{q^{3n+2}}{1 + q^{3n+2}} \right).\end{aligned}$$

Replacing q by $-q$ in (2.4), we have

$$(2.5) \quad \vartheta_3(q)\vartheta_3(q^3) = 1 + 2 \sum_{n=0}^{\infty} \frac{q^{6n+1}}{1 - q^{6n+1}} - \frac{q^{6n+2}}{1 + q^{6n+2}} + \frac{q^{6n+4}}{1 + q^{6n+4}} - \frac{q^{6n+5}}{1 - q^{6n+5}}.$$

Now choosing $z = \frac{\pi}{6}$ in (1.2), we have

$$(2.6) \quad \vartheta_3(q)\vartheta_4(q) \frac{\vartheta_3(\frac{\pi}{6}, q)}{\vartheta_4(\frac{\pi}{6}, q)} = 1 + 4 \prod_{n=1}^{\infty} \frac{q^n \cos \frac{n\pi}{3}}{1 + q^{2n}}.$$

The left-hand side of (2.6) yields

$$\begin{aligned}\prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2 (1 + q^{2n-1} + q^{4n-2})}{(1 + q^{2n})^2 (1 - q^{2n-1} + q^{4n-2})} &= \prod_{n=1}^{\infty} \frac{(1 - q^{6n-3})(1 + q^{2n-1})(1 - q^{2n})^2}{(1 + q^{6n-3})(1 - q^{2n-1})(1 + q^{2n})^2} \\ &= \left(\frac{\vartheta_3^3(q)}{\vartheta_3(q^3)} \right)^{1/2} (\vartheta_4(q)\vartheta_4(q^3))^{1/2}.\end{aligned}$$

Therefore, (2.6) becomes

$$\begin{aligned}
 (2.7) \quad & (\vartheta_4(q)\vartheta_4(q^3))^{1/2} \left(\frac{\vartheta_3^3(q)}{\vartheta_3(q^3)} \right)^{1/2} \\
 &= 1 + 2 \sum_{n=0}^{\infty} (-1)^n \left(\frac{q^{3n+1}}{1+q^{6n+2}} - \frac{q^{3n+2}}{1+q^{6n+4}} + \frac{2q^{3n+3}}{1+q^{6n+6}} \right).
 \end{aligned}$$

Replacement of q by $-q$ gives the Lambert series for

$$(\vartheta_3(q)\vartheta_3(q^3))^{1/2}(\vartheta_4^3(q)/\vartheta_4(q^3))^{1/2}$$

(see (3.20)).

Another possibility of obtaining identities of this kind is to choose $z = \frac{\pi}{3}$ in (1.2). This, however, yields the same identities as that of (2.7) and (3.20). So dnz alone generates six identities pertaining to the modular equation of degree 3. Since there are twelve elliptic functions, it would seem that there will be a total of 72 identities of this nature. But this is not the case. There are many duplications. For example, the replacement of q by $q^{3/2}$ and $z = \frac{\pi}{2} + \frac{\pi\tau}{4}$ in (1.2) is the same as replacing the q in the Fourier series expansion for $nd(\frac{2Kz}{\pi})$ (cf. [7, p. 511]) by $q^{3/2}$ and evaluating it at $z = \frac{\pi\tau}{4}$. It turns out that, by repeating the above method for each of the twelve elliptic functions, there are, in all, 24 such identities which are listed in the next section.

3. THE 24 LAMBERT SERIES PERTAINING TO THE MODULAR EQUATIONS OF DEGREE 3

$$\begin{aligned}
 (3.1) \quad & \frac{\vartheta_2^3(q^{1/2})}{\vartheta_2(q^{3/2})} = 4 \left(1 + 3 \sum_{n=0}^{\infty} \frac{q^{6n+1}}{1-q^{6n+1}} - \frac{q^{6n+5}}{1-q^{6n+5}} \right) \\
 &= 4 \prod_{n=1}^{\infty} \frac{(1-q^{2n})^3}{(1-q^{2n-1})^3} \frac{(1-q^{6n-3})}{(1-q^{6n})} \\
 &= 4 \prod_{n=1}^{\infty} \frac{(1-q^n)^3(1+q^n)^6}{(1-q^{3n})(1+q^{3n})^2} = \frac{1}{2} \left(\frac{9\vartheta_4^3(q^3)}{\vartheta_4(q)} - \frac{\vartheta_4^3(q)}{\vartheta_4(q^3)} \right),
 \end{aligned}$$

$$\begin{aligned}
 (3.2) \quad & \frac{\vartheta_2^3(q^{3/2})}{\vartheta_2(q^{1/2})} = 4 \sum_{n=0}^{\infty} \frac{(q^{2n+1} - q^{4n+2})}{1 - q^{6n+3}} = 4 \sum_{n=0}^{\infty} \left(\frac{q^{3n+1}}{1 - q^{6n+2}} - \frac{q^{3n+2}}{1 - q^{6n+4}} \right) \\
 &= 4q \prod_{n=1}^{\infty} \frac{(1 - q^{3n})^3(1 + q^{3n})^6}{(1 - q^n)(1 + q^n)^2} \\
 &= 2 \left(\frac{\vartheta_4^3(q^6)}{\vartheta_4(q^2)} - \frac{\vartheta_3^3(q^3)}{\vartheta_3(q)} \right) = \frac{1}{2} \left(\frac{\vartheta_4^3(q^3)}{\vartheta_4(q)} - \frac{\vartheta_4^3(q)}{\vartheta_4(q^3)} \right),
 \end{aligned}$$

$$\begin{aligned}
 (3.3) \quad & \frac{\vartheta_3^3(q^3)}{\vartheta_3(q)} = 1 - 2 \sum_{n=0}^{\infty} \left(\frac{q^{6n+1}}{1+q^{6n+1}} - \frac{q^{6n+2}}{1-q^{6n+2}} + \frac{q^{6n+4}}{1-q^{6n+4}} - \frac{q^{6n+5}}{1+q^{6n+5}} \right) \\
 &= \prod_{n=1}^{\infty} \frac{(1 - q^{6n})^3(1 + q^{6n-3})^6}{(1 - q^{2n})(1 + q^{2n-1})^2},
 \end{aligned}$$

$$\begin{aligned}
 (3.4) \quad \frac{\vartheta_3^3(q)}{\vartheta_3(q^3)} &= 1 + 6 \sum_{n=0}^{\infty} \left(\frac{q^{6n+1}}{1 - q^{6n+1}} + \frac{q^{6n+2}}{1 + q^{6n+2}} - \frac{q^{6n+4}}{1 + q^{6n+4}} - \frac{q^{6n+5}}{1 - q^{6n+5}} \right) \\
 &= \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^3 (1 + q^{2n-1})^6}{(1 - q^{6n})(1 + q^{6n-3})^2},
 \end{aligned}$$

$$\begin{aligned}
 (3.5) \quad \frac{\vartheta_4^3(q^3)}{\vartheta_4(q)} &= 1 + 2 \sum_{n=0}^{\infty} \left(\frac{q^{6n+1}}{1 - q^{6n+1}} + \frac{q^{6n+2}}{1 - q^{6n+2}} - \frac{q^{6n+4}}{1 - q^{6n+4}} - \frac{q^{6n+5}}{1 - q^{6n+5}} \right) \\
 &= 1 + 2 \sum_{n=0}^{\infty} (-1)^n \left(\frac{q^{3n+1}}{1 - q^{3n+1}} - \frac{q^{3n+2}}{1 - q^{3n+2}} \right) \\
 &= \prod_{n=1}^{\infty} \frac{(1 - q^{6n})^3 (1 - q^{6n-3})^6}{(1 - q^{2n})(1 - q^{2n-1})^2} = \frac{1}{4} \left(\frac{\vartheta_2^3(q^{1/2})}{\vartheta_2(q^{3/2})} - \frac{\vartheta_2^3(q^{3/2})}{\vartheta_2(q^{1/2})} \right),
 \end{aligned}$$

$$\begin{aligned}
 (3.6) \quad \frac{\vartheta_4^3(q)}{\vartheta_4(q^3)} &= 1 - 6 \sum_{n=0}^{\infty} \left(\frac{q^{3n+1}}{1 + q^{3n+1}} - \frac{q^{3n+2}}{1 + q^{3n+2}} \right) \\
 &= \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^3 (1 - q^{2n-1})^6}{(1 - q^{6n})(1 - q^{6n-3})^2} = \frac{1}{4} \left(\frac{\vartheta_2^3(q^{1/2})}{\vartheta_2(q^{3/2})} - \frac{9\vartheta_2^3(q^{3/2})}{\vartheta_2(q^{1/2})} \right),
 \end{aligned}$$

$$\begin{aligned}
 (3.7) \quad \left(\frac{\vartheta_2^3(q)}{\vartheta_2(q^3)} \right)^{1/2} \left(\frac{\vartheta_4^3(q)}{\vartheta_4(q^3)} \right)^{1/2} &= 2 \left(1 - 3 \sum_{n=0}^{\infty} \frac{q^{6n+1}}{1 + q^{6n+1}} - \frac{q^{6n+5}}{1 + q^{6n+5}} \right) \\
 &= 2 \prod_{n=1}^{\infty} \frac{(1 + q^{6n-3})(1 - q^{2n})^3}{(1 - q^{6n})(1 + q^{2n-1})^3},
 \end{aligned}$$

$$\begin{aligned}
 (3.8) \quad \left(\frac{\vartheta_2^3(q^3)}{\vartheta_2(q)} \right)^{1/2} \left(\frac{\vartheta_4^3(q^3)}{\vartheta_4(q)} \right)^{1/2} &= 2 \sum_{n=1}^{\infty} \frac{q^{2n+1} + q^{4n+2}}{1 + q^{6n+3}} = 2q \prod_{n=1}^{\infty} \frac{(1 - q^{3n})^3 (1 + q^{6n})^3}{(1 - q^n)(1 + q^{2n})} \\
 &= \frac{\vartheta_4^3(q^3)}{\vartheta_4(q)} - \frac{\vartheta_4^3(q^6)}{\vartheta_4(q^2)} = \frac{1}{4} \left(\frac{\vartheta_3^3(q)}{\vartheta_3(q^3)} - \frac{\vartheta_3^3(q^3)}{\vartheta_3(q)} \right),
 \end{aligned}$$

$$\begin{aligned}
 (3.9) \quad \vartheta_2(q)\vartheta_2(q^3) &= 4 \sum_{n=0}^{\infty} \left(\frac{q^{6n+1}}{1 - q^{12n+2}} - \frac{q^{6n+5}}{1 - q^{12n+10}} \right) \\
 &= 4q \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n})^2(1 - q^{6n})(1 + q^{6n})^2 \\
 &= \frac{\vartheta_4^3(q^3)}{\vartheta_4(q)} - \frac{\vartheta_3^3(q^3)}{\vartheta_3(q)} = \frac{1}{3} \left(\frac{\vartheta_3^3(q)}{\vartheta_3(q^3)} - \frac{\vartheta_4^3(q)}{\vartheta_4(q^3)} \right),
 \end{aligned}$$

(3.10)

$$\begin{aligned}
\vartheta_3(q)\vartheta_3(q^3) &= 1 + 2 \sum_{n=1}^{\infty} \left(\frac{q^{6n+1}}{1-q^{6n+1}} - \frac{q^{6n+2}}{1+q^{6n+2}} + \frac{q^{6n+4}}{1+q^{6n+4}} - \frac{q^{6n+5}}{1-q^{6n+5}} \right) \\
&= \prod_{n=1}^{\infty} (1-q^{2n})(1+q^{2n-1})^2(1-q^{6n})(1+q^{6n-3})^2 \\
&= \frac{\vartheta_4^3(q^3)}{\vartheta_4(q)} - \frac{\vartheta_2^3(q^3)}{\vartheta_2(q)} = \frac{1}{3} \left(\frac{\vartheta_2^3(q)}{\vartheta_2^3(q^3)} - \frac{\vartheta_4^3(q)}{\vartheta_4(q^3)} \right),
\end{aligned}$$

(3.11)

$$\begin{aligned}
\vartheta_4(q)\vartheta_4(q^3) &= 1 - 2 \sum_{n=0}^{\infty} (-1)^n \left(\frac{q^{3n+1}}{1+q^{3n+1}} + \frac{q^{3n+2}}{1+q^{3n+2}} \right) \\
&= \prod_1^{\infty} \frac{(1-q^{3n})(1-q^n)}{(1+q^{3n})(1+q^n)} \\
&= \frac{\vartheta_3^3(q^3)}{\vartheta_3(q)} - \frac{\vartheta_2^3(q^3)}{\vartheta_2(q)} = \frac{1}{3} \left(\frac{\vartheta_2^3(q)}{\vartheta_2(q^3)} - \frac{\vartheta_3^3(q)}{\vartheta_3(q^3)} \right),
\end{aligned}$$

(3.12)

$$\begin{aligned}
&(\vartheta_2(q)\vartheta_2(q^3))^{1/2}(\vartheta_4(q)\vartheta_4(q^3))^{1/2} \\
&= 2 \sum_{n=0}^{\infty} \left(\frac{q^{6n+1/2}}{1+q^{12n+1}} - \frac{q^{6n+5/2}}{1+q^{12n+5}} - \frac{q^{6n+7/2}}{1+q^{12n+7}} + \frac{q^{6n+11/2}}{1+q^{12n+11}} \right) \\
&= 2q^{1/2} \prod_1^{\infty} \left(\frac{1-q^{2n}}{1+q^{2n-1}} \right) \left(\frac{1-q^{6n}}{1+q^{6n-3}} \right),
\end{aligned}$$

(3.13)

$$\begin{aligned}
&(\vartheta_2(q)\vartheta_2(q^3))^{1/2} \left(\frac{\vartheta_3^3(q^3)}{\vartheta_3(q)} \right)^{1/2} = 2 \sum_{n=0}^{\infty} \frac{q^{3n+1/2}}{1+q^{6n+1}} \\
&= 2q^{1/2} \prod_1^{\infty} \frac{(1-q^{6n})^2(1+q^{6n-3})(1+q^{2n})}{(1-q^{6n-3})^2(1+q^{6n})(1+q^{2n-1})} \\
&= \frac{1}{2}(\vartheta_2^2(q) - \vartheta_2^2(q^3)),
\end{aligned}$$

(3.14)

$$\begin{aligned}
&(\vartheta_2(q)\vartheta_2(q^3))^{1/2} \left(\frac{\vartheta_3^3(q)}{\vartheta_3(q^3)} \right)^{1/2} = 2 \sum_{n=0}^{\infty} \left(\frac{q^{3n+1/2}}{1-q^{6n+1}} + \frac{2q^{3n+3/2}}{1-q^{6n+3}} + \frac{q^{3n+5/2}}{1-q^{6n+5}} \right) \\
&= 2q^{1/2} \prod_{n=1}^{\infty} \frac{(1+q^{6n})(1+q^{2n-1})(1-q^{2n})^2}{(1+q^{6n-3})(1+q^{2n})(1-q^{2n-1})^2} \\
&= \frac{1}{2}(\vartheta_2^2(q) + 3\vartheta_2^2(q^3)),
\end{aligned}$$

(3.15)

$$\begin{aligned}
 (\vartheta_2(q)\vartheta_2(q^3))^{1/2} \left(\frac{\vartheta_4^3(q^3)}{\vartheta_4(q)} \right)^{1/2} &= 2 \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n+1/2}}{1 - q^{6n+1}} \\
 &= 2q^{1/2} \prod_{n=1}^{\infty} \frac{(1 - q^{6n})^2 (1 - q^{6n-3}) (1 + q^{2n})}{(1 + q^{6n-3})^2 (1 + q^{6n}) (1 - q^{2n-1})} \\
 &= \frac{1}{2} (\vartheta_2^2(q) + \vartheta_2^2(q^3)),
 \end{aligned}$$

(3.16)

$$\begin{aligned}
 (\vartheta_2(q)\vartheta_2(q^3))^{1/2} \left(\frac{\vartheta_4^3(q)}{\vartheta_4(q^3)} \right)^{1/2} &= 2 \sum_{n=0}^{\infty} \left(\frac{q^{3n+1/2}}{1 + q^{6n+1}} - \frac{2q^{3n+3/2}}{1 + q^{6n+3}} + \frac{q^{3n+5/2}}{1 + q^{6n+5}} \right) \\
 &= 2q^{1/2} \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2 (1 - q^{2n-1}) (1 + q^{6n})}{(1 + q^{2n-1})^2 (1 + q^{2n}) (1 - q^{6n-3})} \\
 &= \frac{1}{2} (\vartheta_2^2(q) - 3\vartheta_2^3(q^3)),
 \end{aligned}$$

(3.17)

$$\begin{aligned}
 (\vartheta_3(q)\vartheta_3(q^3))^{1/2} \left(\frac{\vartheta_2^3(q^3)}{\vartheta_2(q)} \right)^{1/2} &= 2 \sum_{n=0}^{\infty} (-1)^n \left(\frac{q^{2n+1} + q^{4n+2}}{1 - q^{6n+3}} \right) \\
 &= 2 \sum_{n=0}^{\infty} \left(\frac{q^{3n+1}}{1 + q^{6n+2}} + \frac{q^{3n+2}}{1 + q^{6n+4}} \right) \\
 &= 2q \prod_{n=1}^{\infty} \frac{(1 + q^{2n-1}) (1 + q^{6n}) (1 - q^{6n})^2}{(1 + q^{2n}) (1 + q^{6n-3}) (1 - q^{6n-3})^2} \\
 &= \frac{1}{2} (\vartheta_3^2(q) - \vartheta_3^2(q^3)),
 \end{aligned}$$

$$\begin{aligned}
 (\vartheta_3(q)\vartheta_3(q^3))^{1/2} \left(\frac{\vartheta_2^3(q)}{\vartheta_2(q^3)} \right)^{1/2} &= 2 \left(1 + \sum_{n=0}^{\infty} (-1)^n \left(\frac{q^{6n+1}}{1 - q^{6n+1}} + \frac{2q^{6n+3}}{1 - q^{6n+3}} + \frac{q^{6n+5}}{1 - q^{6n+5}} \right) \right) \\
 &= 2 \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2 (1 + q^{2n}) (1 + q^{6n-3})}{(1 - q^{2n-1})^2 (1 + q^{2n-1}) (1 + q^{6n})} \\
 &= \frac{1}{2} (\vartheta_3^2(q) + 3\vartheta_3^2(q^3)),
 \end{aligned}$$

(3.19)

$$\begin{aligned}
 (\vartheta_3(q)\vartheta_3(q^3))^{1/2} \left(\frac{\vartheta_4^3(q^3)}{\vartheta_4(q)} \right)^{1/2} &= 1 + 2 \sum_{n=1}^{\infty} \frac{(q^n + q^{5n})}{1 + q^{6n}} \\
 &= \prod_1^{\infty} \left(\frac{1 + q^{2n-1}}{1 - q^{2n-1}} \right) \frac{(1 - q^{6n-3}) (1 - q^{6n})^2}{(1 + q^{6n-3}) (1 + q^{6n})^2} \\
 &= \frac{1}{2} (\vartheta_3^2(q) + \vartheta_3^2(q^3)),
 \end{aligned}$$

(3.20)

$$\begin{aligned}
 (\vartheta_3(q)\vartheta_3(q^3))^{1/2} \left(\frac{\vartheta_4^3(q)}{\vartheta_4(q^3)} \right)^{1/2} &= 1 - 2 \sum_{n=0}^{\infty} \left(\frac{q^{3n+1}}{1+q^{6n+2}} + \frac{q^{3n+2}}{1+q^{6n+4}} - \frac{2q^{3n+3}}{1+q^{6n+6}} \right) \\
 &= \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2(1-q^{2n-1})(1+q^{6n-3})}{(1+q^{2n})^2(1+q^{2n-1})(1-q^{6n-3})} \\
 &= \frac{1}{2}(3\vartheta_3^2(q^3) - \vartheta_3^2(q)),
 \end{aligned}$$

(3.21)

$$\begin{aligned}
 (\vartheta_4(q)\vartheta_4(q^3))^{1/2} \left(\frac{\vartheta_2^3(q^3)}{\vartheta_2(q)} \right)^{1/2} &= 2 \sum_{n=0}^{\infty} (-1)^n \frac{(q^{2n+1} - q^{4n+2})}{1+q^{6n+3}} \\
 &= 2q \prod_{n=1}^{\infty} \frac{(1-q^{2n-1})(1+q^{6n})(1-q^{6n})^2}{(1+q^{2n})(1-q^{6n-3})(1+q^{6n-3})^2} \\
 &= \frac{1}{2}(\vartheta_4^2(q^3) - \vartheta_4^2(q)),
 \end{aligned}$$

(3.22)

$$\begin{aligned}
 (\vartheta_4(q)\vartheta_4(q^3))^{1/2} \left(\frac{\vartheta_2^3(q)}{\vartheta_2(q^3)} \right)^{1/2} &= 2 \left(1 - \sum_{n=0}^{\infty} (-1)^n \left(\frac{q^{6n+1}}{1+q^{6n+1}} + \frac{2q^{6n+3}}{1+q^{6n+3}} + \frac{q^{6n+5}}{1+q^{6n+5}} \right) \right) \\
 &= 2 \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2(1+q^{2n})(1-q^{6n-3})}{(1+q^{2n-1})^2(1-q^{2n-1})(1+q^{6n})} \\
 &= \frac{1}{2}(\vartheta_4^2(q) + 3\vartheta_4^2(q^3)),
 \end{aligned}$$

(3.23)

$$\begin{aligned}
 (\vartheta_4(q)\vartheta_4(q^3))^{1/2} \left(\frac{\vartheta_3^3(q^3)}{\vartheta_3(q)} \right)^{1/2} &= 1 + 2 \sum_{n=1}^{\infty} (-1)^n \left(\frac{q^n + q^{5n}}{1+q^{6n}} \right) \\
 &= \prod_{n=1}^{\infty} \frac{(1-q^{2n-1})(1+q^{6n-3})(1-q^{6n})^2}{(1+q^{2n-1})(1-q^{6n-3})(1+q^{6n})^2} \\
 &= \frac{1}{2}(\vartheta_4^2(q) + \vartheta_4^2(q^3)),
 \end{aligned}$$

(3.24)

$$\begin{aligned}
 (\vartheta_4(q)\vartheta_4(q^3))^{1/2} \left(\frac{\vartheta_3^3(q)}{\vartheta_3(q^3)} \right)^{1/2} &= 1 + 2 \sum_{n=0}^{\infty} (-1)^n \left(\frac{q^{3n+1}}{1+q^{6n+2}} - \frac{q^{3n+2}}{1+q^{6n+4}} + \frac{2q^{3n+3}}{1+q^{6n+6}} \right) \\
 &= \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2(1+q^{2n-1})(1-q^{6n-3})}{(1+q^{2n})^2(1-q^{2n-1})(1+q^{6n-3})} \\
 &= \frac{1}{2}(3\vartheta_4^2(q^3) - \vartheta_4^2(q)).
 \end{aligned}$$

Remarks. 1. Using the well-known facts that $\vartheta_3(q) = \vartheta_4(-q)$ and $2\vartheta_2(q)\vartheta_3(q) = \vartheta_2^2(q^{1/2})$, one sees easily that the identities (3.1) and (3.7) are derivable from each other by replacing q by $-q$. So is the pair (3.3) and (3.5). There are twelve such pairs; so out of these 24 identities, only 12 of them are genuinely distinct. This list is even smaller if one further takes the imaginary transformation [7, p. 474] for the theta functions into account.

2. Some of the modular identities such as

$$\frac{\vartheta_2^3(q^{3/2})}{\vartheta_2(q^{1/2})} = 2 \left(\frac{\vartheta_4^3(q^6)}{\vartheta_4(q^2)} - \frac{\vartheta_3^3(q^3)}{\vartheta_3(q)} \right),$$

$$\vartheta_2(q)\vartheta_2(q^3) = \frac{\vartheta_4^3(q^3)}{\vartheta_4(q)} - \frac{\vartheta_3^3(q^3)}{\vartheta_3(q)}$$

listed in (3.2) and (3.9) are evident from their Lambert series representations. The less obvious ones such as

$$\frac{\vartheta_2^3(q^{3/2})}{\vartheta_2(q^{1/2})} = \frac{1}{2} \left(\frac{\vartheta_4^3(q^3)}{\vartheta_4(q)} - \frac{\vartheta_4^3(q)}{\vartheta_4(q^3)} \right)$$

listed also in (3.2) will be derived in the following section.

4. MODULAR IDENTITIES

With the collection of these identities, many modular identities become easy to deduce. For example, from (1.3) and (3.19), we see that

$$\begin{aligned} \vartheta_3^2(q) + \vartheta_3^2(q^3) &= 2 + 4 \sum_{n=1}^{\infty} \left(\frac{q^n}{1+q^{2n}} + \frac{q^{3n}}{1+q^{6n}} \right) \\ &= 2 \left(1 + 2 \sum_{n=1}^{\infty} \frac{q^n(1-q^{2n}+q^{4n})+q^{3n}}{1+q^{6n}} \right) \\ (4.1) \quad &= 2 \left(1 + 2 \sum_{n=1}^{\infty} \frac{q^n+q^{5n}}{1+q^{6n}} \right) \\ &= 2(\vartheta_3(q)\vartheta_3(q^3))^{1/2} \left(\frac{\vartheta_4^3(q^3)}{\vartheta_4(q)} \right)^{1/2}. \end{aligned}$$

And, from (1.3) and (3.17),

$$\begin{aligned} \vartheta_3^2(q) - \vartheta_3^2(q^3) &= 4 \sum_{n=1}^{\infty} \left(\frac{q^n}{1+q^{2n}} - \frac{q^{3n}}{1+q^{6n}} \right) \\ (4.2) \quad &= 4 \sum_{n=1}^{\infty} \left(\frac{q^{3n+1}}{1+q^{6n+2}} + \frac{q^{3n+2}}{1+q^{6n+4}} \right) \\ &= 2(\vartheta_3(q)\vartheta_3(q^3))^{1/2} \left(\frac{\vartheta_2^3(q^3)}{\vartheta_2(q)} \right)^{1/2}. \end{aligned}$$

The identity (4.2) is well known (see [6, p. 216, (12.5.7)]) and is frequently written as $\frac{1}{m} = 1 + 2(l^3/k)^{1/4}$ where $m = \vartheta_3^2(q^3)/\vartheta_3^2(q)$, $l = \vartheta_2^2(q^3)/\vartheta_3^2(q^3)$, and $k = \vartheta_2^2(q)/\vartheta_3^2(q)$.

Multiplying (3.21) and (3.23), we have

$$\vartheta_4^4(q^3) - \vartheta_4^4(q) = 2\vartheta_4(q)\vartheta_4(q^3)\frac{\vartheta_2^3(q^{3/2})}{\vartheta_2(q^{1/2})}.$$

Hence

$$(4.3) \quad \frac{\vartheta_4^3(q^3)}{\vartheta_4(q)} - \frac{\vartheta_4^3(q)}{\vartheta_4(q^3)} = \frac{2\vartheta_2^3(q^{3/2})}{\vartheta_2(q^{1/2})}.$$

Similarly, the product of (3.22) and (3.24) yields

$$(4.4) \quad \frac{9\vartheta_4^3(q^3)}{\vartheta_4(q)} - \frac{\vartheta_4^3(q)}{\vartheta_4(q^3)} = \frac{2\vartheta_2^3(q^{1/2})}{\vartheta_2(q^{3/2})}.$$

More identities of this nature can be obtained. For example, multiplying (3.13) and (3.14), we get

$$(4.5) \quad 4\vartheta_2(q)\vartheta_2(q^3)\vartheta_3(q)\vartheta_3(q^3) = \vartheta_2^4(q) - 3\vartheta_2^4(q^3) + 2\vartheta_2^2(q)\vartheta_2^2(q^3),$$

and multiplying (3.15) and (3.16), we have

$$(4.6) \quad 4\vartheta_2(q)\vartheta_2(q^3)\vartheta_4(q)\vartheta_4(q^3) = \vartheta_2^4(q) - 3\vartheta_2^4(q^3) - 2\vartheta_2^2(q)\vartheta_2^2(q^3).$$

Adding (4.5) and (4.6) together, we obtain

$$(4.7) \quad \vartheta_3(q)\vartheta_3(q^3) + \vartheta_4(q)\vartheta_4(q^3) = \frac{1}{2} \frac{\vartheta_2^3(q)}{\vartheta_2(q^3)} - \frac{3}{2} \frac{\vartheta_2^3(q^3)}{\vartheta_2(q)};$$

and subtraction of (4.6) from (4.5) gives the familiar modular equation of degree 3,

$$(4.8) \quad \vartheta_3(q)\vartheta_3(q^3) - \vartheta_4(q)\vartheta_4(q^3) = \vartheta_2(q)\vartheta_2(q^3).$$

Similarly, from (3.17)–(3.20) and (3.21)–(3.24), we obtain, respectively,

$$(4.9) \quad \vartheta_2(q)\vartheta_2(q^3) - \vartheta_4(q)\vartheta_4(q^3) = \frac{1}{2} \frac{\vartheta_3^3(q)}{\vartheta_3(q^3)} - \frac{3}{2} \frac{\vartheta_3^3(q^3)}{\vartheta_3(q)}$$

and

$$(4.10) \quad \begin{aligned} & \vartheta_2(q)\vartheta_2(q^3) + \vartheta_3(q)\vartheta_3(q^3) \\ &= \frac{3}{2} \frac{\vartheta_4^3(q^3)}{\vartheta_4(q)} - \frac{1}{2} \frac{\vartheta_4^3(q)}{\vartheta_4(q^3)} = \frac{1}{4} \frac{\vartheta_2^3(q^{1/2})}{\vartheta_2(q^{3/2})} + \frac{3}{4} \frac{\vartheta_2^3(q^{3/2})}{\vartheta_2(q^{1/2})}. \end{aligned}$$

The second identity of (4.10) follows immediately from (4.3) and (4.4).

We now establish that (4.10) also can be expressed in terms of the Lambert series as follows:

$$(4.11) \quad \frac{1}{4} \frac{\vartheta_2^3(q^{1/2})}{\vartheta_2(q^{3/2})} + \frac{3}{4} \frac{\vartheta_2^3(q^{3/2})}{\vartheta_2(q^{1/2})} = 1 + 6 \sum_{n=0}^{\infty} \left(\frac{q^{3n+1}}{1 - q^{3n+1}} - \frac{q^{3n+2}}{1 - q^{3n+2}} \right).$$

This is done by using the Lambert series in (3.1) and (3.2). First, we observe that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{q^{2n+1}}{1 - q^{6n+3}} &= \sum_{n=1}^{\infty} \frac{q^n}{1 - q^{3n}} - \frac{q^{2n}}{1 - q^{6n}} = \sum_{n=1}^{\infty} \frac{q^n - q^{2n} + q^{4n}}{1 - q^{6n}}, \\ \sum_{n=0}^{\infty} \frac{q^{4n+2}}{1 - q^{6n+3}} &= \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - q^{3n}} - \frac{q^{4n}}{1 - q^{6n}} = \sum_{n=1}^{\infty} \frac{q^{2n} - q^{4n} + q^{5n}}{1 - q^{6n}}. \end{aligned}$$

Hence from (3.2)

$$(4.12) \quad \frac{1}{4} \frac{\vartheta_2^3(q^{3/2})}{\vartheta_2(q^{1/2})} = \sum_{n=0}^{\infty} \frac{q^{2n+1} - q^{4n+2}}{1 - q^{6n+3}} = \sum_{n=1}^{\infty} \frac{q^n - 2q^{2n} + 2q^{4n} - q^{5n}}{1 - q^{6n}} \\ = \sum_{n=0}^{\infty} \frac{q^{6n+1}}{1 - q^{6n+1}} - \frac{2q^{6n+2}}{1 - q^{6n+2}} + \frac{2q^{6n+4}}{1 - q^{6n+4}} - \frac{q^{6n+5}}{1 - q^{6n+5}}.$$

Clearly, the identity (4.11) follows immediately from (3.1) and (4.12). And incidentally, we also note that

$$(4.13) \quad \frac{1}{4} \frac{\vartheta_2^3(q^{1/2})}{\vartheta_2(q^{3/2})} - \frac{3}{4} \frac{\vartheta_2^3(q^{3/2})}{\vartheta_2(q^{1/2})} = 1 + 6 \sum_{n=0}^{\infty} \left(\frac{q^{6n+2}}{1 - q^{6n+2}} - \frac{q^{6n+4}}{1 - q^{6n+4}} \right).$$

Remarks. In [5] J. M. and P. B. Borewein and Garvan introduce

$$a(q) = \sum_{m, n=-\infty}^{\infty} q^{n^2+mn+m^2}$$

to study certain cubic modular identities of Ramanujan. In particular, they prove, using a different method, that

$$a(q) = \vartheta_2(q)\vartheta_2(q^3) + \vartheta_3(q)\vartheta_3(q^3) \\ = 1 + 6 \sum_{n=0}^{\infty} \left(\frac{q^{3n+1}}{1 - q^{3n+1}} - \frac{q^{3n+2}}{1 - q^{3n+2}} \right) \\ = \frac{3}{2} \frac{\vartheta_4^3(q^3)}{\vartheta_4(q)} - \frac{1}{2} \frac{\vartheta_4^3(q)}{\vartheta_4(q^3)}.$$

To obtain additional identities of a different kind, we mention the following set of identities:

$$(4.14) \quad \vartheta_2(q) - \vartheta_2(q^9) = 2q^{1/4} \prod_{n=1}^{\infty} (1 - q^{18n})(1 + q^{18n-6})(1 + q^{8n-12}),$$

$$(4.15) \quad \vartheta_3(q) - \vartheta_3(q^9) = 2q \prod_{n=1}^{\infty} (1 - q^{18n})(1 + q^{18-3})(1 + q^{18n-15}),$$

$$(4.16) \quad \vartheta_4(q) - \vartheta_4(q^9) = -2q \prod_{n=1}^{\infty} (1 - q^{18n})(1 - q^{18n-3})(1 - q^{18n-15}),$$

$$(4.14)' \quad \vartheta_4(q) - 3\vartheta_4(q^9) = -2 \prod_{n=1}^{\infty} \frac{(1 - q^{2n})(1 + q^n)^2}{(1 + q^{3n})},$$

$$(4.15)' \quad \vartheta_3(q) - 3\vartheta_3(q^9) = -2 \prod_{n=1}^{\infty} \frac{(1 - q^{2n})(1 + q^{6n-3})}{(1 + q^{2n-1})},$$

$$(4.16)' \quad \vartheta_2(q) - 3\vartheta_2(q^9) = 2q^{1/4} \prod_{n=1}^{\infty} \frac{(1 - q^{2n})(1 + q^{6n})}{(1 + q^{2n})}.$$

We remark that (4.14)', (4.15)', and (4.16)' are obtained directly from (4.14), (4.15), and (4.16) by the familiar imaginary transformation of the theta func-

tions. The proof of (4.14) is elementary and is as follows:

$$\begin{aligned}
 \vartheta_2(q) - \vartheta_2(q^9) &= \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} - q^{(3n+3/2)^2} \\
 &= \sum_{n=-\infty}^{\infty} (q^{(3n-1/2)^2} + q^{(3n+1/2)^2} + q^{(3n+3/2)^2}) - q^{(3n+3/2)^2} \\
 &= 2q^{1/4} \sum_{n=-\infty}^{\infty} q^{9n^2} q^{3n} \\
 &= 2q^{1/4} \prod_{n=1}^{\infty} (1 - q^{18n})(1 + q^{18n-6})(1 + q^{18n-12}).
 \end{aligned}$$

The proof (4.15) is similar, we omit it here.

It is interesting to observe that from (3.11) and (4.16) we have

$$\begin{aligned}
 \sum_{n=0}^{\infty} (-1)^n &\left\{ \frac{q^{9n+1}}{1+q^{9n+1}} + \frac{q^{9n+2}}{1+q^{9n+2}} - \frac{q^{9n+3}}{1+q^{9n+3}} + \frac{q^{9n+4}}{1+q^{9n+4}} \right. \\
 &\quad \left. + \frac{q^{9n+5}}{1+q^{9n+5}} - \frac{q^{9n+6}}{1+q^{9n+6}} + \frac{q^{9n+7}}{1+q^{9n+7}} + \frac{q^{9n+8}}{1+q^{9n+8}} \right\} \\
 &= \frac{1}{2} (\vartheta_4(q^3) \vartheta_4(q^9) - \vartheta_4(q) \vartheta_4(q^3)) = \frac{\vartheta_4(q^3)}{2} (\vartheta_4(q) - \vartheta_4(q^9)) \\
 &= q \prod_{n=1}^{\infty} \frac{(1 - q^{3n})(1 - q^{18n})}{(1 + q^{3n})^2 (1 - q^{18n-9})}.
 \end{aligned}$$

We now derive the following beautiful identities of Ramanujan:

$$(4.17) \quad \left(\frac{\vartheta_i(q)}{\vartheta_i(q^9)} - 1 \right)^3 = \left(\frac{\vartheta_i(q^3)}{\vartheta_i(q^9)} \right)^4 - 1,$$

$$(4.17)' \quad \left(\frac{3\vartheta_i(q^9)}{\vartheta_i(q)} - 1 \right)^3 = \frac{9\vartheta_i^4(q^3)}{\vartheta_i^4(q)} - 1$$

where $i = 2, 3, 4$.

Since the proofs of these six identities are identical, we will only do (4.17) for $i = 2$.

Multiplying (3.13) and (3.15), we obtain

$$\vartheta_2^4(q) - \vartheta_2^4(q^3) = 16q \prod_{n=1}^{\infty} \frac{(1 - q^{6n})^4 (1 + q^{2n})^3}{(1 + q^{6n})}.$$

Hence

$$\left(\frac{\vartheta_2(q)}{\vartheta_2(q^3)} \right)^4 - 1 = \frac{\vartheta_2^4(q) - \vartheta_2^4(q^3)}{\vartheta_2^4(q^3)} = \frac{1}{q^2} \prod_{n=1}^{\infty} \frac{(1 + q^{2n})^3}{(1 + q^{6n})^9}.$$

On the other hand, from (4.14), we have

$$\frac{\vartheta_2(q)}{\vartheta_2(q^9)} - 1 = \frac{1}{q^2} \prod_{n=1}^{\infty} \frac{(1 + q^{6n})}{(1 + q^{18n})^3}.$$

Clearly, (4.17) now follows immediately from the above two identities.

It is, perhaps, worthwhile to point out that the product of (4.14)–(4.16) gives the following identity (see [3, p. 147]):

$$\left(1 - \frac{\vartheta_2(q)}{\vartheta_2(q^9)}\right) \left(1 - \frac{\vartheta_3(q)}{\vartheta_3(q^9)}\right) \left(1 - \frac{\vartheta_4(q)}{\vartheta_4(q^9)}\right) = 4.$$

5. CONCLUDING REMARKS

We note that some additional identities can still be derived from (1.1). One of the most interesting identities is

$$\begin{aligned} (5.1) \quad q^{1/3} \prod_{n=1}^{\infty} \frac{(1 - q^{3n})^3}{(1 - q^n)} &= \sum_{n=0}^{\infty} \left(\frac{q^{n+1/3}}{1 - q^{3n+1}} - \frac{q^{2n+4/3}}{1 - q^{3n+2}} \right) \\ &= \sum_{k, n=0}^{\infty} q^{(3n+1)(3k+1)/3} - q^{(3n+2)(3k+2)/3} \\ &= \sum_{n=0}^{\infty} (d_{1,3}(3n+1) - d_{2,3}(3n+1)) q^{n+1/3}, \end{aligned}$$

where $d_{i,3}(n)$ denotes the number of divisors of n which are congruent to $i \pmod{3}$, $i = 1, 2$.

To prove (5.1), we choose $y = -\frac{x}{2}$ in (1.1). Then it becomes

$$\sum_{n=-\infty}^{\infty} \frac{e^{2inx}}{1 - e^{-ix}q^{2n}} = -\frac{i}{2} \frac{\vartheta'_1}{\vartheta_1(x)}.$$

Now replacing q by $q^{3/2}$ and choosing $x = \pi\tau$ in the above identity, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{q^{n+1}}{1 - q^{3n+1}} - \frac{q^{2n+2}}{1 - q^{3n+2}} \\ = - \sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1 - q^{3n-1}} = \frac{i}{2} \frac{\vartheta'_1(0, q^{3/2})}{\vartheta_1(\pi\tau, q^{3/2})} = q \prod_{n=1}^{\infty} \frac{(1 - q^{3n})^3}{(1 - q^n)}. \end{aligned}$$

This establishes the first part of the identity in (5.1), the remaining two identities on the right-hand side of (5.1) are elementary and we omit their proofs.

Many and, perhaps, all of the identities that appear in this paper are well known and scattered throughout the vast identities literature. The methods of deriving them, however, are very different. For example, the identities (4.14)–(4.16) and (4.14)'–(4.16)' are established in [3, p. 143], the proofs of (3.1) and (3.4) can be found in [2], and all the proofs involve the use of the quintuple product identity. The identity (5.1) is proved in [4, p. 697] using the function $a(q)$ defined in the remarks of §4. In this note, we see that they can be derived from the single identity (1.1).

To end our discussion, we quote the following statement by Professor Berndt from the introduction of his recent *Ramanujan's notebook*. III:

“Although we provide proofs here for all of Ramanujan's formulas, in many cases, we have been unable to find the roads that led Ramanujan to his discoveries. It is hoped that others will attempt to discover the pathways that Ramanujan took on his journey through his luxuriant labyrinthine forest of enchanting and alluring formulas.”

It is quite conceivable that Ramanujan, with the possession of very general identities such as ${}_1\psi_1$, had methodically carried out the evaluation of these identities at all the conceivable values and thus compiled an endless list of identities. But it is truly the genius of Ramanujan which penetrated and grasped completely the intricate inter-relations among these identities, which, in turn, gives rise to this multitude of modular identities.

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