PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 122, Number 4, December 1994

THE EXISTENCE OF BOUNDED INFINITE DTr-ORBITS

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(Communicated by Ken Goodearl)

ABSTRACT. We construct an indecomposable module over a symmetric algebra whose DTr-orbit is infinite and bounded. This yields a counterexample to a conjecture which states that the number of modules in an Auslander-Reiten component having the same length is finite.

Let Λ be an Artin algebra, $\mathscr{C}$ a connected component of the Auslander-Reiten quiver of Λ, and DTr the Auslander-Reiten translation [1]. In [8], Ringel asked whether the number of modules having the same length in $\mathscr{C}$ is always finite. This is the case when Λ is a hereditary algebra [2, 10] or a tame algebra [4]. For an arbitrary algebra Λ, the question has an affirmative answer if $\mathscr{C}$ has at most finitely many nonperiodic DTr-orbits [3, 6] or is a regular component of the form $\mathbf{Z}\Delta$ with Δ one of A_∞^∞, B_∞, C_∞, or D_∞ [7].

The aim of this paper is to show that the above problem has no affirmative answer in general. We shall construct a local symmetric algebra whose Auslander-Reiten quiver contains a bounded infinite DTr-orbit. Our example will be a modification of that given by the second author in a different context [9].

Let K be a field which contains an element ρ of infinite multiplicative order. Let R be the polynomial ring over K in noncommutating variables X and Y modulo the ideal generated by X^2, Y^2, and $YX - \rho XY$. Then R is a local Frobenius algebra over K with radical $J(R) = xR + yR$, $J(R)^2 = \mathrm{Soc}(R) = xyR$, and $J(R)^3 = 0$, where x, y denote the residue classes of X, Y, respectively. Let $DR = \mathrm{Hom}_K(R, K)$ be the dual of R with the following R-R-bimodule structure: given r', $r'' \in R$ and $f \in DR$, $r'fr''$ is the K-linear map which sends $r \in R$ to $f(r''rr')$. Let T be the trivial extension algebra of R by DR which is the K-vector space $T = R \oplus DR$ with multiplication given by

$$(r, f)(r', f') = (rr', rf' + fr')$$

for $r, r' \in R$ and $f, f' \in DR$. Then T is a local symmetric K-algebra with radical

$$J(T) = \{(r, f) | r \in J(R), f \in DR\}$$

Received by the editors March 25, 1993.
1991 *Mathematics Subject Classification*. Primary 16G10, 16G70.
Key words and phrases. Artin algebra, Auslander-Reiten quiver, DTr-orbit.

and $J(T)^4 = 0$. We choose a K-basis $1, x, y, xy, a, b, c, d$ of T, where a, b, c, d is the K-basis of DR dual to the K-basis $1, x, y, xy$ of R. We find the following multiplication table of T:

	1	x	y	xy	a	b	c	d
1	1	x	y	xy	a	b	c	d
x	x	0	xy	0	0	a	0	ρc
y	y	ρxy	0	0	0	0	a	b
xy	xy	0	0	0	0	0	0	a
a	a	0	0	0	0	0	0	0
b	b	a	0	0	0	0	0	0
c	c	0	a	0	0	0	0	0
d	d	c	ρb	a	0	0	0	0

For a right T-module N, let ΩN denote the kernel of a minimal projective cover of N. It is well known that $\Omega^2 N = DTrN$, since T is a symmetric algebra. Let now M be the module $M = (x+y)T$. We shall compute $\Omega^i M$ for all $i \in \mathbf{Z}$. Since M is the image of the map from T to T given by the left multiplication with the element $x+y$, the module ΩM is the right annihilator of $x+y$ in T. Using the equation

$$(x+y)(A+Bx+Cy+Dxy+Ea+Fb+Gc+Hd) = 0,$$

where $A, B, C, D, E, F, G, H \in K$, one finds $A = H = 0$, $C + B\rho = 0$, and $F + G = 0$. Hence, $\Omega M = (x+(-\rho)y)T + (-b+c)T$. Note that $(x+(-\rho)y)d = \rho(-b+c)$. This implies that $\Omega M = (x+(-\rho)y)T$. By induction, one can show that $\Omega^i M = (x+(-\rho)^i y)T$ for all $i \in \mathbf{Z}$. Using the fact that $T = K + J(T)$ and $J(T)^4 = 0$, one has

$$M(x+(-\rho)^2 y)J(T) = (x+y)T(x+(-\rho)^2 y)J(T) = (x+y)(x+(-\rho)^2 y)J(T) \neq 0$$

and

$$\Omega M(x+(-\rho)^2 y)J(T) = (x+(-\rho)y)(x+(-\rho)^2 y)J(T) = 0.$$

Hence, M and ΩM have different annihilators in T. Therefore, they are not isomorphic. Using the fact that ρ is not a root of unity, one can similarly show that the module $\Omega^i M$ is not annihilated by $(x+(-\rho)^{j+1}y)J(T)$ for $j \neq i$, whereas $\Omega^j M$ is. Thus the modules $\Omega^i M$ are pairwise nonisomorphic. Obviously, $\dim_K \Omega^i M = 4$ for all $i \in \mathbf{Z}$. Consequently all modules in the DTr-orbits of M have the same dimension over K.

Remarks. (1) We conjecture that a stable component of an Auslander-Reiten quiver is of the form $\mathbf{Z}A_\infty$ if it contains infinitely many modules of the same length. It has been shown in [6] that this is the case if one of its DTr-orbits contains infinitely many modules of the same length.

(2) In [9, §4], the R-module $M = (x+y)R$ has been used to give an example of an Ω-bounded but not Ω-periodic module. The following computation shows that $DTrM \cong M$ in this case. The sequence

$$R \xrightarrow{(x+(-\rho)y)\bullet} R \xrightarrow{(x+y)\bullet} M \to 0$$

induces a sequence

$$R \xrightarrow{\bullet(x+(-\rho)y)} R \xrightarrow{\bullet(x+(-\rho)^2 y)} TrM \to 0;$$

hence, $DTrM = \mathrm{Hom}_K(R(x + (-\rho)^2 y), K)$. This right R-module is annihilated by $x + (-\rho)y$ and hence, is isomorphic to $(x+y)R$. Note that the last argument does not work if one replaces R with T.

(3) Another example of an infinite DTr-orbit of bounded modules can be constructed by using Example 3.2 in [5].

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