SURFACES MINIMIZING AREA PLUS LENGTH OF SINGULAR CURVES

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ABSTRACT. We give the first existence and regularity results on surfaces minimizing area plus length of singular curves, as in energy-minimizing interfaces in materials.

1. INTRODUCTION

Interfaces in materials are often modeled by area-minimizing surfaces, such as the soap film on tetrahedral frame of Figure 1.1(a) (on the next page). Such surfaces typically meet in threes at 120° angles along singular curves. A better model might add a cost ("disclination energy") proportional to the length of the singular curves. [MT] suggests that such a minimizer of area plus length might look like the surface of Figure 1.1(b), with more area but a shorter singular curve. Note that *four* pieces of surface meet along the short curve in the center. Rough estimates suggest that this phenomenon would occur at a scale just below what can be detected by today's technology.

This paper gives the first mathematical results on the existence and structure of such energy-minimizing surfaces. The results apply to physical examples and provide a model for computer approximations (see Brakke [Br2]).

Earlier work by Cook [C] minimized area plus "length" for a single piece of surface with a free boundary "curve" constrained to lie in a plane, in higher dimensions. Recently we have discovered a number of the methods of our paper in excellent earlier work of Ecker [E] on the so-called thread problem, a problem in many ways more difficult, although it considers just a single surface (see §5). After completing this work, we found a conjecture by De Giorgi [DG, Congettura 2.7] on the existence of certain surfaces minimizing area plus length of singular curves.

More generally, there has been much work in various contexts on minimizing sums of energies of different dimensions, although typically codimensions 0 and 1. For example, work of Mumford and Shah [MuS] helps a computer eye find the curves delineating images by minimizing an energy involving area and

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FIGURE 1.1. The area-minimizing soap film (a) models interfaces in materials. If an energy cost is added for the singular curves, surface (b) becomes cheaper than (a). Note that four pieces of surface meet along the short curve in the center. (Photo by F. Goro. Computer picture by J. Taylor.)

length in a planar domain. Our surfaces in space present completely different problems.

1.1. The main results. Following Brakke (cf. [Br1]), our existence and regularity results (Theorem 2.1) require an a priori prescription of the combinatorial structure of the solution, including the number of singular curves and which pieces of surface they bound. The theorem concludes that each singular curve $C_i - \partial C_i$ is smoothly embedded, that the associated pieces of surface are smooth minimal submanifolds with boundary along $C_i - \partial C_i$, and that their inward conormals $n_1, n_2, \dots$ and the curvature vector κ of the curve satisfy the balancing condition

$$(4) \quad n_1 + n_2 + \cdots + \kappa = 0.$$

Unfortunately, the theorem theoretically allows unanticipated intersections and crossings of surfaces impossible for interfaces between regions.

We conjecture that without any combinatorial prescriptions, the solution is a smoothly embedded stratified manifold: minimal surfaces meet in threes or fours along curves, which in turn meet only in threes at isolated points (see Conjecture 3.1). That surfaces meet in threes or fours is suggested by recent work of the Williams College “SMALL” undergraduate research Geometry Group [CCF] on the analogous lower-dimensional problem of networks minimizing length plus the number of added nodes or “Steiner points”. In such networks segments sometimes meet in threes or fours but never in fives or more.

1.2. Surfaces prescribed to meet ν at a time. For surfaces meeting ν at a time for a fixed integer ν , there is a simpler approach: minimize $\text{area}(S) + \frac{1}{\nu} \text{length}(\partial S - B)$ among integral currents with $\partial S \equiv B \pmod{\nu}$. This boundary condition means that

$$\partial S - B = \nu C$$

for some “singular curve” C . Existence follows from the compactness theorem for surfaces modulo ν (see [M2, 11.1]). In this context, known regularity says that the surface is embedded.

1.3. The thread problem. Section 5 describes some implications for the problem of finding a least-area surface or soap film bounded by a piece of wire and a piece of thread.

1.4. Edge-length-minimizing polyhedra. Another question involving the length of singular curves goes back to Fejes-Toth [FT, p. 305]:

Find the polyhedron of unit volume of least total edge length.

Melzak [Me, (B), p. 567] conjectured the equilateral right triangular prism. The data from Berger [Be], as compiled in Table 1, compares some of the leading contenders.

Raising the stakes from the 1991 Mt. Holyoke Regional Geometry Institute, I now offer \$200 for a solution. Meanwhile, I offer a copy of my book [M2] for finding an example that beats the triangular prism or for proving that a minimizer exists (even among *convex* polyhedra). The problem with existence is that infinitely many short edges may appear in the limit.

TABLE 1

<i>Polyhedron (unit volume)</i>	<i>Length</i>
Triangular prism	11.896
Cube	12.000
Tetrahedron	12.238
Pentagonal prism	12.518
Square pyramid	12.814
Triangular dipyrmaid	14.342
Dodecahedron	15.217
Octahedron	15.419
Icosahedron	23.131

2. EXISTENCE AND REGULARITY THEOREM

The following theorem, given fixed boundary curves $C_1, \dots, C_m$, produces surfaces $S_1, \dots, S_{m_2}$, free boundary curves $C_{m+1}, \dots, C_{m_1}$, and points $P_1, \dots, P_{m_0}$, meeting in prescribed ways, minimizing $\sum \text{area}(S_i) + \sum_{j>m} \text{length}(C_j)$. “Surface” technically means *flat chain modulo two*, thus allowing singularities, nonorientability, and unrestricted topological type (see [M2, Chapters 11, 1, 4, 5]).

2.1. Theorem. Existence and regularity with prescribed combinatorial structure. In $\mathbb{R}^3$ let $C_1, \dots, C_m$ be rectifiable curves. Fix integers $m_0 \geq 0$, $m_1 \geq m$, $m_2 \geq 0$. Let I_i, J_i be fixed sets of integer indices. If there are any surfaces (flat chains modulo two) $S_1, \dots, S_{m_2}$, rectifiable curves $C_{m+1}, \dots, C_{m_1}$, and points $P_1, \dots, P_{m_0}$ such that

$$(1) \quad \partial S_i = \sum_{j \in I_i} C_j \quad (1 \leq i \leq m_2),$$

$$(2) \quad \partial C_i = \sum_{j \in J_i} P_j \quad (1 \leq i \leq m_1)$$

(modulo two), then they may be chosen to minimize

$$(3) \quad \sum \text{area}(S_i) + \sum_{i>m} \text{length}(C_i).$$

Each $S_i - \bigcup_{j \leq m} C_j - \bigcup P_j$ is a smooth, embedded minimal surface on the interior. Each singular curve $C_j - \partial C_j$ ($j > m$) is a $C^{1,1}$ embedded curve. At any nonboundary point in a single C_j , if $j > m$, each S_i is a smooth (C^∞) embedded manifold with boundary; the curvature of C_j does not exceed the number of surfaces S_i it bounds by prescription (1), and the inward conormals $n_1, n_2, \dots$ of these surfaces and the curvature vector κ of C_i sum to 0:

$$(4) \quad n_1 + n_2 + \dots + \kappa = 0.$$

If P_j bounds only free boundary curves by prescription (2), i.e., if $j \in J_i$ only if $i > m$, the onward unit tangents to these curves at P_j sum to 0.

2.2. Example. One interesting example consists of $m \geq 3$ half-discs S_i bounded by m semicircles C_i and a common diameter C_{m+1} , with the conormals summing to 0. (Here $m_0 = 2$, $m_1 = m + 1$, $m_2 = m$, $I_i = \{i, m + 1\}$, each $J_i = \{1, 2\}$, and all the C_i have the same endpoints P_1, P_2 .) This example separately minimizes both $\sum \text{area}(S_i)$ and of course $\text{length } C_{m+1}$. It minimizes area by a standard slicing argument (cf. [M2, Lemma 10.6]). The slice itself is stationary because the m vectors by hypothesis sum to 0. Since length is a convex function of the position of the point where the m vectors meet, the slice is the length-minimizing way to connect the m given points to a free point.

The surface on the tetrahedral frame of Figure 1.1(b) is presumably another example.

2.3. Remark. The theorem and proof immediately generalize to minimize

$$(3') \quad \sum \lambda_i \text{area}(S_i) + \sum_{i>m} \mu_i \text{length}(C_i)$$

for positive weights λ_i , μ_i . Of course, the terms in the balancing condition (4) need to carry the associated weights.

2.4. Proof of Theorem. Since $\text{area}(S_i)$, $\text{length}(C_i)$, $\text{length}(\partial S_i)$, and $\text{card}(\partial C_i)$ are bounded by (3), (3), (1), and (2), respectively, the existence of a minimizing collection follows by compactness (see [M2, 11.1, 5.5, 5.6] or [F1, (4.2.17), p. 432]).

Of course, each S_i must be area minimizing for its boundary, hence a smooth, embedded minimal surface ([R] or [F2]).

Suppose p lies in a single C_j (but not in ∂C_j) and $j > m$. A first variation argument shows that weakly the curvature of C_j ($j > m$) is at most the number of surfaces it bounds by prescription (1):

$$|\kappa| \leq \kappa_j \equiv \text{card}\{i : i \in I_j\}.$$

Otherwise smooth deformations of C_j decreasing length (leaving the surfaces alone), coupled with the addition of small patches of surface, would decrease area plus length. This argument depends on the fact that the solution is minimizing, not merely stationary.

Therefore, C_j is a $C^{1,1}$ immersion with Lipschitz constant on the derivative of an arclength parameterization at most κ_j . Since the surfaces are unoriented, C_j must be embedded by a simple comparison argument.

Since the surfaces are unoriented, each associated S_i is a $C^{1,\alpha}$ manifold with boundary along C_j at p ($0 < \alpha < 1$) ([All1, p. 523], proved in [All2, §5]). Furthermore, if C_j is $C^{k,\alpha}$, so is each S_i , by the standard Schauder theory [GT, Theorem 6.19]. The vanishing of the first variation of (3) yields (4) (weakly, since C_j is not yet proved C^2).

Now by (4), since the conormals are C^α , κ is C^α ; hence, C_j is $C^{2,\alpha}$; hence, each S_i is $C^{2,\alpha}$; hence, the conormals are $C^{1,\alpha}$; hence, κ is $C^{1,\alpha}$; hence, C_j is $C^{3,\alpha}$; and so on. We conclude that C_j is C^∞ and that the surfaces meet smoothly along C_j at p .

The final balancing condition at the singular points is the standard equilibrium condition for length-minimization, which applies here because at a small scale the area effects are negligible.

3. A CONJECTURE

The following conjecture removes the a priori restrictions on the combinatorial structure of Theorem 2.1 and gives stronger regularity. Regions are prescribed outside the unit ball. Singular curves are allowed to escape penalty by moving off to the boundary, as perhaps the material attempts to form layers.

3.1. Conjecture. Existence and regularity. Let $R_1, \dots, R_m$ be measurable partitioning subsets of $\mathbf{B}(0, 2) \subset \mathbf{R}^3$ prescribed in $\mathbf{B}(0, 2) - \mathbf{B}(0, 1)$. Let S denote the interface between regions, restricted to $\mathbf{B}(0, 1)$; i.e., $S = \bigcup \partial R_i \cap \mathbf{B}(0, 1)$. Then $R_1, \dots, R_m$ may be chosen to minimize $\text{area } S + \mathcal{H}^1(\text{sing } S)$. Here $\text{reg } S$ and $\text{sing } S$ denote the portions of S inside the open unit ball where S is or is not a C^1 embedded manifold. $\text{reg } S$ is a smooth (C^∞) minimal surface. $\text{sing } S$ consists of smooth curves, along which three or four pieces of surface meet smoothly, in turn meeting smoothly in threes at 120° at isolated points. Along the singular curves, the inward conormals n_i and the curvature

vector κ of the singular curve sum to 0:

$$\sum n_i + \kappa = 0.$$

At the singular points, all pieces of surface have a common tangent plane.

3.2. Remarks. For similar “soap film” problems minimizing area alone, existence follows by viewing the R_i as integral currents of geometric measure theory and minimizing $\frac{1}{2} \sum \mathbf{M}(\partial R_i)$. Work of Taylor [T] and Almgren [Alm] shows that minimizers consist of smooth surfaces meeting in threes at 120° angles along curves, which in turn meet in fours at angles of about 109° .

The evidence for 3.1 that surfaces meet in threes and fours includes the tetrahedral example of Figure 1.1 and the lower-dimensional fact that certain networks minimizing length plus the number of Steiner points meet in threes and fours [CCF]. Five or more surfaces meeting could be stable but not minimizing.

The singular curves meet in threes because length-minimizing networks meet in threes. At such a singular point, the three curves determine the plane tangent to the pieces of surface.

The proof seems beyond current technology. The main obstacle in proving the conjectured existence is that additional singular curves could appear in the limit; i.e., $\mathcal{H}^1(\text{sing } S)$ may not be lower semicontinuous. The main difficulty in proving regularity is that neither area nor length alone is minimized. On a small scale the singular curve should be nearly minimal, *unless* shortening it would create other singular curves.

We conjecture similar existence and regularity for the harder problem of regions of prescribed volumes minimizing area plus singular length. See the upcoming [M1].

4. HIGHER DIMENSION AND CODIMENSION

The existence results of Theorem 2.1 generalize to collections of surfaces S_i^d of dimension $0 \leq d \leq m$ in $\mathbf{R}^n$ minimizing

$$\sum_{d=1}^m \sum_i \text{area}(S_i^d).$$

The m -dimensional surfaces are area minimizing and hence embedded minimal surfaces (away from their boundaries) except for an interior singular set of dimension at most $m - 2$ [F2, Theorem 2, p. 771]. The lower-dimensional surfaces are $(\mathbf{M}, \varepsilon, \delta)$ -minimal and hence embedded C^1 manifolds almost everywhere [Alm, IV.13(6)].

Conjecture 3.1 extends to hypersurfaces S in $\mathbf{R}^n$ minimizing

$$\mathcal{H}^{n-1}(S) + \mathcal{H}^{n-2}(\text{sing } S).$$

$\text{Reg } S$ is an embedded minimal hypersurface. $\text{Sing } S$ consists of $(n - 2)$ -dimensional surfaces along which three or four minimal hypersurfaces meet, $(n - 3)$ -dimensional surfaces along which three $(n - 2)$ -dimensional surfaces meet, $(n - 4)$ -dimensional surfaces along which four $(n - 3)$ -dimensional surfaces meet, and lower-dimensional singular sets. ($\text{Sing } S$ itself should have soap-film-like singularities, which are not yet classified for films of dimension $n - 2 > 2$. See [T, W1, W2].)

5. THE THREAD PROBLEM

For convenience consider a single surface S_1 . Any minimizer of

$$\text{area } S_1 + \sum_{j>m} \text{length } C_j$$

as before (or more generally $\text{area } S_1 + \lambda \sum_{j>m} \text{length } C_j$ for some $\lambda > 0$) is automatically a minimizer of $\text{area } S_1$ given $\text{length } C_j$. This second problem may be interpreted as attaching fixed lengths of movable thread to fixed boundary curves $C_1, \dots, C_m$ and seeking the least-area surface ("soap film") with that partially free boundary. See Ecker [E] or, for the classical mapping setting, Alt [A], Hildebrandt [H1, H2], and Nitsche [N]. Our conclusion (4) implies that the free boundary is an embedded curve of constant curvature 1 (more generally $1/\lambda$); cf. [E, Proposition 2.7], whose requirement that the surface be oriented means that the free boundary curve need not be embedded.

A solution to the second "thread" problem of minimizing $\text{area } S_1$ given $\text{length } C_j$ need not be a solution to the first problem of minimizing $\text{area } S_1 + \lambda \sum \text{length } C_j$ for any Lagrange multiplier λ . For example, take a single fixed boundary curve C_1 to be a large arc of a circle of curvature $1/2$ as in Figure 5.1. Taking the single free boundary curve C_1 to be a large arc of a circle of curvature 1 provides the unique solution to the thread problem for its length, but taking $C_2 = C_1$ (and $S_1 = \emptyset$) minimizes $\text{area } S_1 + \text{length } C_2$ (indeed minimizes $\text{area } S_1 + \lambda \text{length } C_2$ for all $\lambda \leq 2$).

Actual soap films have interior singularities that complicate the problem even for a single fixed boundary curve closed by a single thread. For a boundary curve that is $(3, 1)$ torus "knot", there is a soap film consisting of three strips meeting at 120° along a singular curve. If a piece of the boundary is replaced by a suitably long piece of thread, the center of the thread will bound two strips of surface, and its curvature, determined by 2.1(4), will vary. See Figure 5.2. This soap film could be modeled by a version of Theorem 2.1 using surfaces modulo three.

FIGURE 5.1. C_2 is the best thread for its length, but taking $C_2 = C_1$ and $S_1 = \emptyset$ minimizes $\text{area } S_1 + \text{length } C_2$.

FIGURE 5.2. With actual soap films, if the thread extends through an interior singular curve, it will not have constant curvature. (b) shows a piece of a soap film bounded by the (3,1) torus knot of (a). A thread replacing a segment of boundary as in (c) will not have constant curvature.

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