ON THE HYPERBOLICITY AND THE SCHOTTKY PROPERTY OF COMPLEX SPACES

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ABSTRACT. The purpose of this paper is to study the hyperbolicity and the tautness of spaces that have the Schottky property. Moreover, the hyperbolicity of compact complex spaces is characterized by the classical theorem of Bloch.

INTRODUCTION

In [4] Hahn and Kim considered complex manifolds satisfying the Schottky property and claimed that a complex manifold M is hyperbolic if and only if M has the Schottky property (see [4, Theorem 1]). However, that theorem is not true.

In addition, the authors stated that a compact complex manifold M is hyperbolic if and only if every map f in $\text{Hol}(D, M)$ is Bloch (see [4, Theorem 2]). However, their proof of this theorem uses a lemma for which they gave an incorrect proof (see [4, §3]).

The aims of this paper are to present counterexamples to Theorem 1 in [4] and to give a correct proof of Theorem 2 in [4].

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1. COMPLEX SPACES SATISFYING THE SCHOTTKY PROPERTY

We first give the following

1.1. Definition. Let M be a complex space and h_M a hermitian metric on the tangent space TM of M . Assume that ρ_M is the distance function (or the integrated metric) associated with h_M . The space M is said to have the Schottky property for h_M if for each $p \in M$, each relatively compact open set W in a coordinate neighbourhood of p , and each $r \in (0, 1)$, there exists a positive constant $S = S(W, r)$ such that every map $f \in \text{Hol}(D, M)$ with $f(0) \in W$ satisfies $\rho_M(p, f(z)) \leq S$ for $|z| \leq r$, where D is the open unit disc in $\mathbb{C}$ and $\text{Hol}(D, M)$ denotes the space of holomorphic maps from D into M equipped with the compact-open topology.

We now present counterexamples to Theorem 1 in [4].

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1.2. Counterexamples. (1) By the continuity of distance functions, every compact complex space M satisfies the Schottky property for any hermitian metric h_M on TM . Taking M to be complex projective space, we see that having the Schottky property does not imply that M is hyperbolic.

(ii) Let M be a bounded domain in $\mathbb{C}^n$ that is not a domain of holomorphy, and let h_M be a complete hermitian metric on TM .

Then M is hyperbolic, but M does not satisfy the Schottky property for h_M .

Indeed, under the assumption that M satisfies the Schottky property, we will prove that $\text{Hol}(D, M)$ is a normal family. This will mean that M is taut [1, Theorem 2, p. 430] and hence, a domain of holomorphy [10, Theorem 2, pp. 211–212], which is a contradiction.

Let $\{f_n\}$ be a sequence in $\text{Hol}(D, M)$ that is not compactly divergent. By definition this means that there are compact subsets K of D and K' of M such that $f_n(K) \cap K' \neq \emptyset$ for infinitely many n . Thus there are a subsequence $\{f_{n_k}\}$ and points $z_k \in K$ such that $f_{n_k}(z_k) \in K'$ for all k . We can assume that the sequence $\{z_k\}$ converges to a point $z_0 \in K$ and the sequence $\{f_{n_k}(z_k)\}$ converges to a point $p \in K$. For $k = 0, 1, 2, \dots$ set

$$\beta_k(z) = \frac{z + z_k}{1 + \overline{z}_k z}.$$

Then [8, p. 13] β_k is a holomorphic automorphism of D and β_k converges to β_0 as $k \rightarrow \infty$. Put $g_k = f_{n_k} \circ \beta_k$ for all $k \geq 1$. Since the sequence $\{g_k(0)\} = \{f_{n_k}(z_k)\}$ converges to p , we can assume that there is a relatively compact open set W in a coordinate neighbourhood of p such that $g_k(0) \in W$ for all $k \geq 1$. For each fixed $z \in D$, the Schottky property implies that the sequence $\{g_k(z)\}$ is contained in the compact ball $\{x \in M: \rho_M(p, x) \leq S(W, |z|)\}$. Now $\{g_k\}$ is obviously equicontinuous with respect to the Kobayashi distance d_M , which defines the underlying topology on M (see [2] or [8, Theorem I.2.3, pp. 19–20]). Hence by the Ascoli-Arzelà theorem we may assume that $\{g_k\}$ converges to a map g in $\text{Hol}(D, M)$. Then the subsequence $\{f_{n_k}\}$ converges to the map $g \circ \beta_0^{-1}$ in $\text{Hol}(D, M)$. Thus $\text{Hol}(D, M)$ is a normal family, as claimed.

1.3. Remark. The domain M in counterexample (ii) has the Schottky property for the hermitian metric induced by the canonical metric of $\mathbb{C}^n$. This shows that the Schottky property of a complex space can depend on the choice of the hermitian metric.

By an analogous proof as in Counterexample 1.2(ii), we have the following

1.4. Proposition. *Let M be a hyperbolic complex space and h_M a complete hermitian metric on TM . Then M is taut if and only if M has the Schottky property for h_M .*

2. THE HYPERBOLICITY OF COMPACT COMPLEX SPACES

First we recall the following

2.1. Definition (see [4]). Let M be a complex space and h_M a hermitian metric on TM . A map $f \in \text{Hol}(D, M)$ is said to be Bloch if

$$\sup\{(1 - |z|^2) \cdot h_M(f(z), f'(z)): z \in D\} < \infty.$$

We now prove the following (see [4, Theorem 2])

2.2. Theorem. *A compact complex space M is hyperbolic if and only if every map $f \in \text{Hol}(D, M)$ is Bloch.*

Proof. ($\Rightarrow$) Assume that M is hyperbolic. By [7, Theorem 8.1, p. 378], $\text{Hol}(D, M)$ is compact; thus $R = \sup\{h_M(g(0), g'(0)): g \in \text{Hol}(D, M)\} < \infty$. Let $f \in \text{Hol}(D, M)$ and $z \in D$. Set

$$\beta(w) = \frac{w + z}{1 + \bar{z}w}.$$

Then [8, p. 13] β is a holomorphic automorphism of D , so $g = f \circ \beta \in \text{Hol}(D, M)$. The chain rule gives

$$R \geq h_M(g(0), g'(0)) = (1 - |z|^2) \cdot h_M(f(z), f'(z)),$$

which shows that f is Bloch.

($\Leftarrow$) Let $f: \mathbb{C} \rightarrow M$ be a holomorphic map. Assume that $p, q \in \mathbb{C}$. Consider two sequences $\{z_n\}, \{z'_n\}$ of D such that $z_m \neq z'_n$ for all $m, n \geq 1$, $\lim_{n \rightarrow \infty} z_n = \lim_{n \rightarrow \infty} z'_n = 1$, and $\lim_{n \rightarrow \infty} d_D(z_n, z'_n) = 0$. By a theorem of Weierstrass, there exists a holomorphic map $\gamma: D \rightarrow \mathbb{C}$ such that $\gamma(z_n) = p$, $\gamma(z'_n) = q$ for all $n \geq 1$. Put $g = f \circ \gamma \in \text{Hol}(D, M)$. Since g is Bloch, there exists a positive constant $c_g > 0$ such that

$$(*) \quad h_M(g(z), g'(z)) \leq c_g / (1 - |z|^2) \quad \text{for all } z \in D.$$

By integrating both sides of $(*)$ along the geodesic curve between two z_n and z'_n in D and using the definition of the integrated distance ρ_M , we have $\rho_M(g(z_n), g(z'_n)) \leq c_g \cdot d_D(z_n, z'_n)$ for all $n \geq 1$. Hence $\rho_M(f(p), f(q)) = 0$ or $f(p) = f(q)$. Thus M contains no nontrivial complex lines. By a theorem of Brody (see [3] or [8, Theorem 2.1, p. 68]), M is hyperbolic. $\square$

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